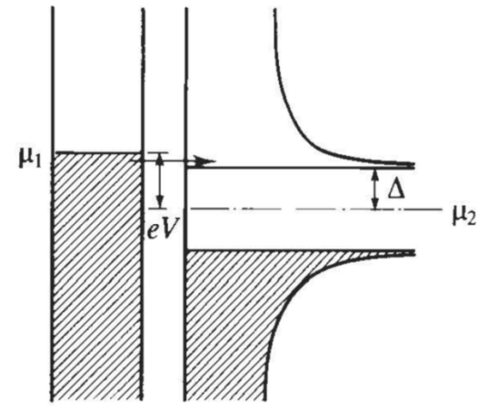
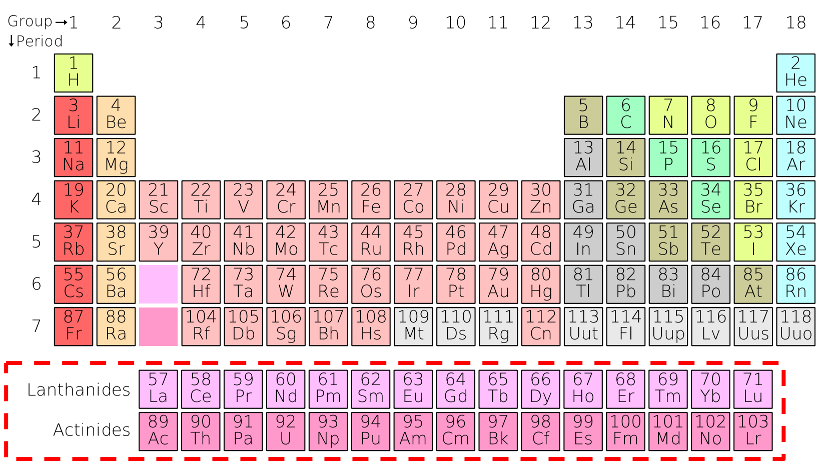
Last time: BCS and heavy-fermion superconductors

- **Bardeen-Cooper Schrieffer (conventional) superconductors**
  - Discovered in **1911** by Kamerlingh-Onnes
  - Fully gapped Bogoliubov quasiparticle spectrum
  - Important effects
    - Vanishing resistivity
    - Meissner effect (London penetration depth)
    - Coherence effects (coherence length)



- **Heavy-fermion superconductors**
  - Discovered by Steglich *et al*. in **1979**
  - Key ingredients
    - **Lattice** of *f*-electrons
    - Conduction electrons
  - Multiple superconducting phases

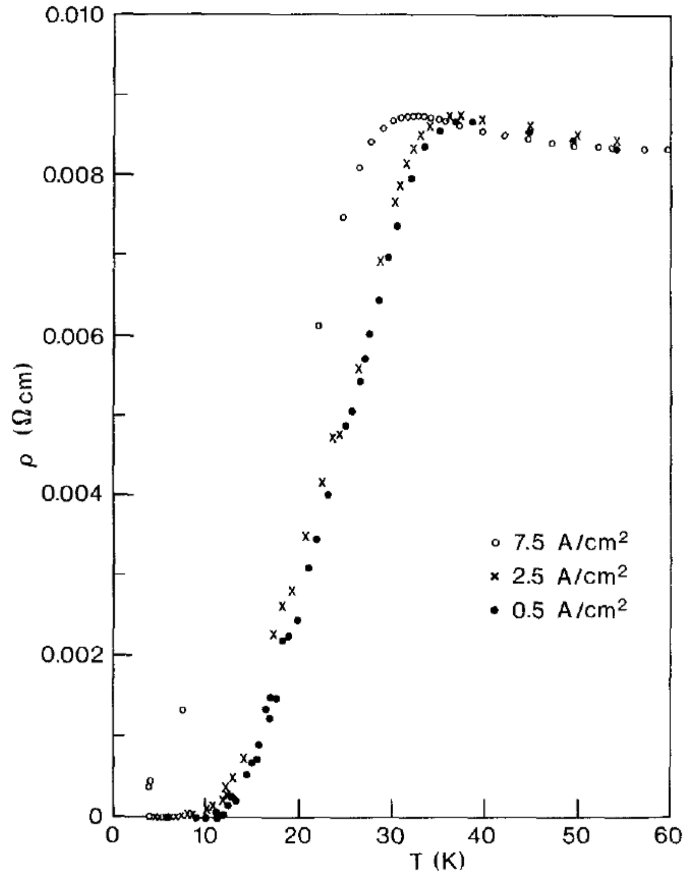


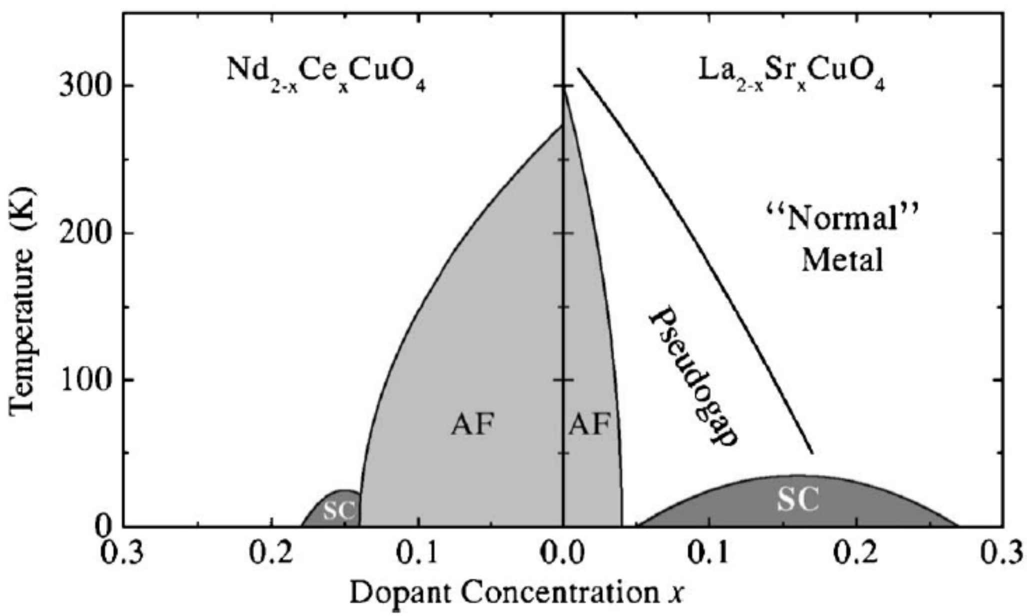
Cuprates overview

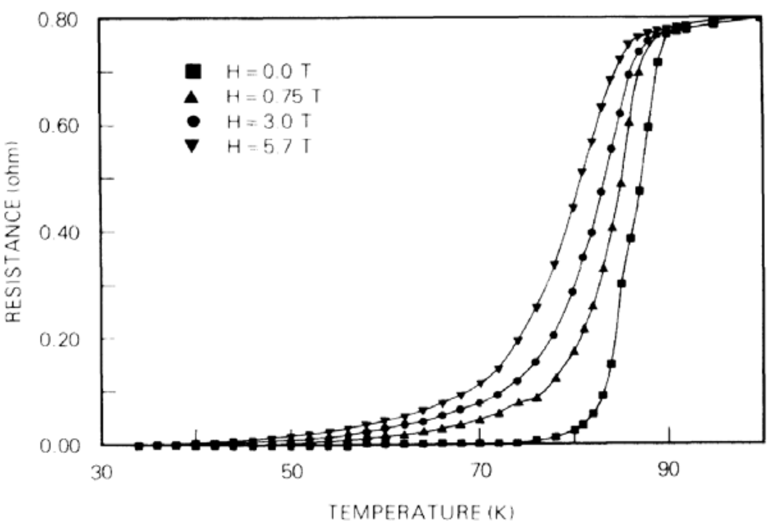
- **Introduction and Phenomenology**
- **Experiments**
  - Pseudogap
  - Stripes
  - Nodal quasiparticles
- **Introduction to Resonating Valence Bond (RVB)**
- **Phase fluctuations vs. competing order**
- **Numerical techniques**
- **Single hole problem**
- **Slave particles and gauge fields**
  - Mean field theory
  - U(1) gauge theory
  - Confinement physics

I. Introduction (cuprates)

- **Discovery**
  - Bednorz and Müller reported $T_c \approx 30$ K in Ba-doped La_2CuO_4 in **1986**
  - Highest BCS superconductor was Nb_3Ge with $T_c = 23.2$ K
  - N_2 barrier ➔ $T_c > 77$ K in YBCO
  - "Universal" phase diagram

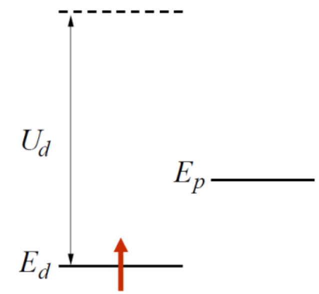


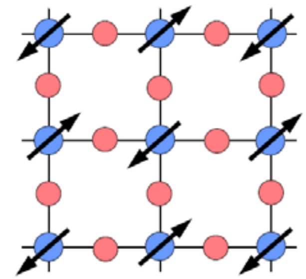


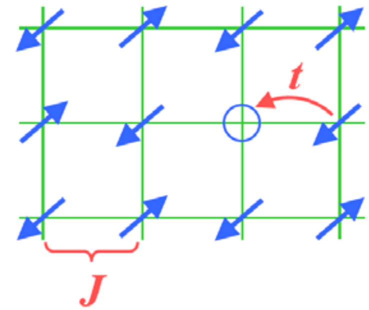


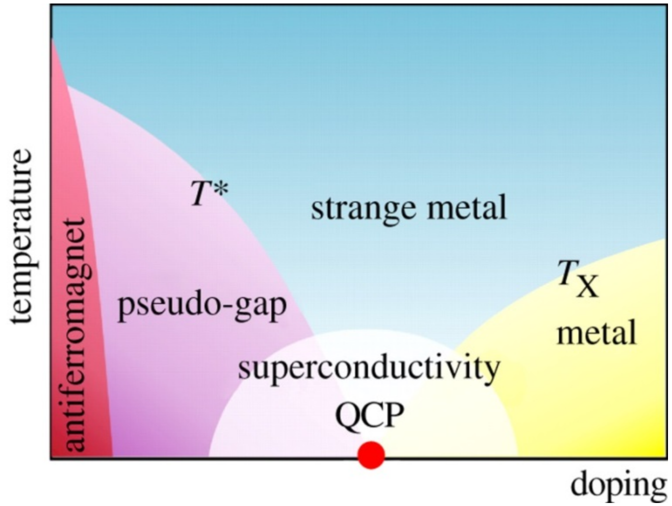
II. Basic electronic structure of the cuprates

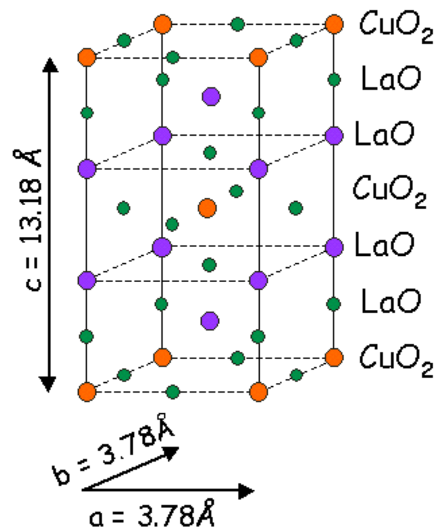
- **Lattice, bonding, and doping**
  - Relevant energy scales:
    - t ➔ hopping energy
    - U_d ➔ double-occupancy penalty
  - La_2CuO_4: La^{3+}, Cu^{2+}, O^{4-}; 1 hole doped by $La^{3+} \rightarrow Sr^{2+}$
  - $La_{1.85}Sr_{0.15}CuO_4$ (LSCO) ➔ $T_c \approx 40$ K
  - $YBa_2Cu_3O_7$: Y^{3+}, Ba^{2+}, Cu^{2+}, O^{4-}; already hole doped!
  - $YBa_2Cu_3O_{7-\varepsilon}$ (YBCO) ➔ $T_c \approx 93$ K

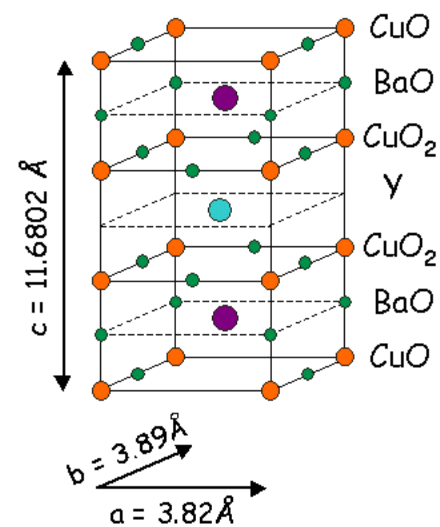










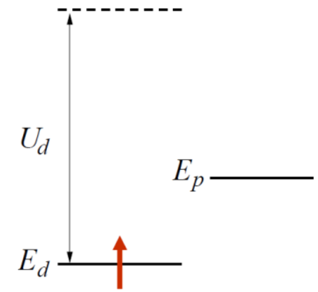


II. Basic electronic structure of the cuprates

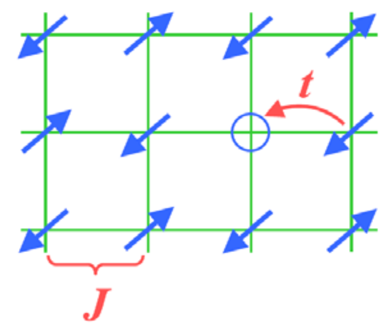
- **Theoretical modeling**
  - The "*t-J* model" Hamiltonian

$$H = P\left[-\sum_{\langle \mathbf{ij}\rangle,\sigma} t_{\mathbf{ij}} c^{\dagger}_{\mathbf{i}\sigma} c_{\mathbf{i}\sigma} + J\sum_{\langle \mathbf{ij}\rangle}\left(\mathbf{S_i}\cdot\mathbf{S_j} - \frac{1}{2}n_{\mathbf{i}}n_{\mathbf{j}}\right)\right]P$$

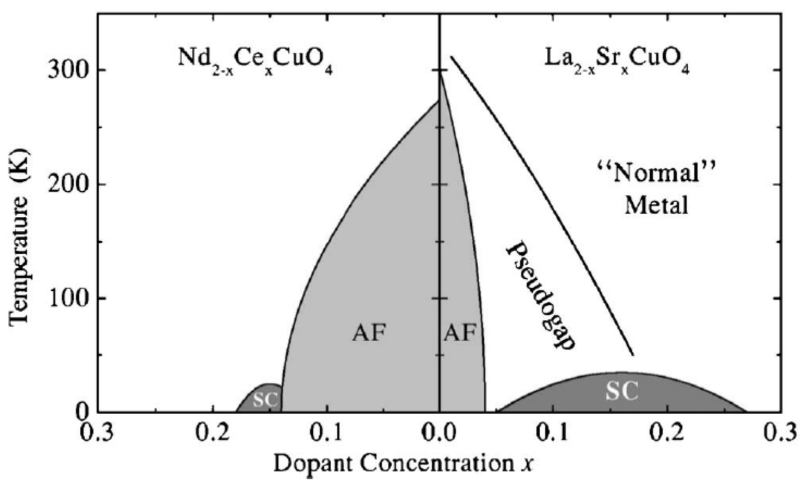
$$J = \frac{t_{pd}^4}{(E_p - E_d)^3}$$



  - Projection operator P restricts the Hilbert space to one which excludes double occupation of any site



  - Next-nearest (t') and next-next-nearest (t'') hopping gives better fits to data
  - A non-zero t' accounts for asymmetry in electron and hole doped systems
  - *Weak* coupling between CuO_2 layers gives non-zero T_c
  - Cuprates are "quasi-2D" ➔ 2D layer describes the entire phase diagram



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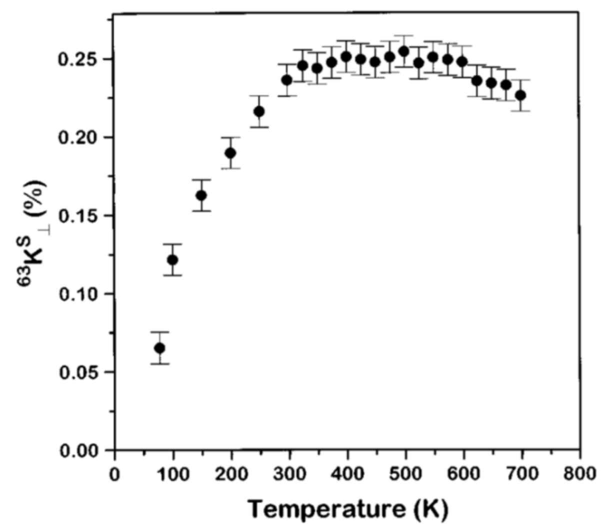
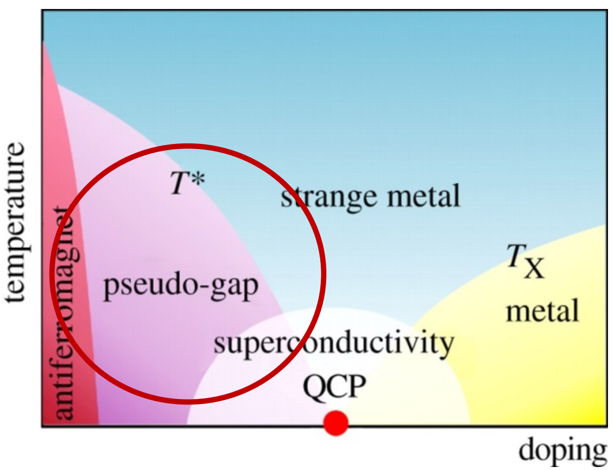
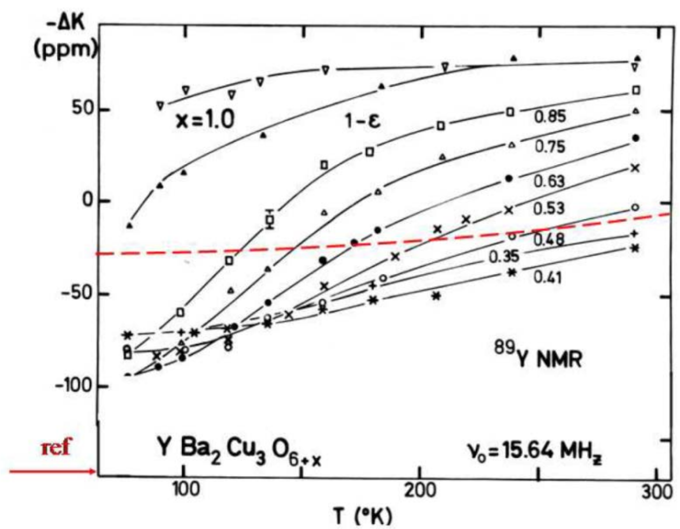
III. Phenomenology of the underdoped cuprates

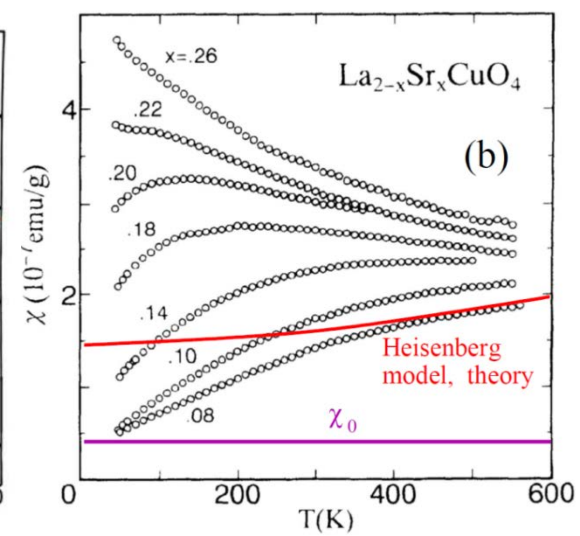
A. The pseudogap phenomenon in the normal state

- **Magnetic properties**
  - NMR/Knight shift on YBCO (T_c = 79 K)
  - χ_s is T-independent from 300 K to 700 K
  - χ_s drops below Heisenberg model expectation before T_c
  - Strongly points to singlet formation as origin of pseudogap





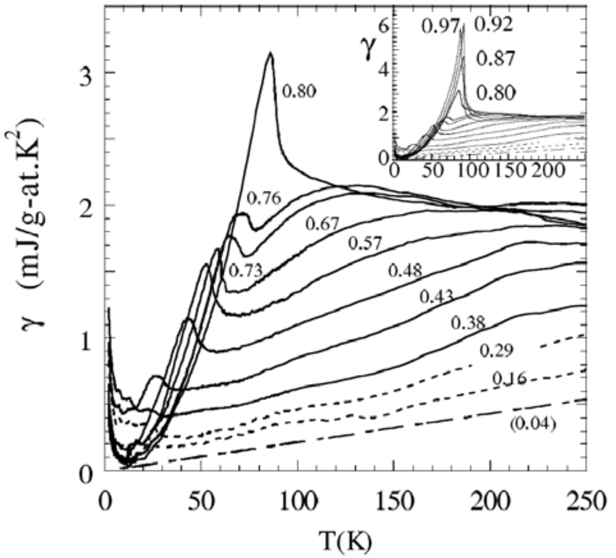


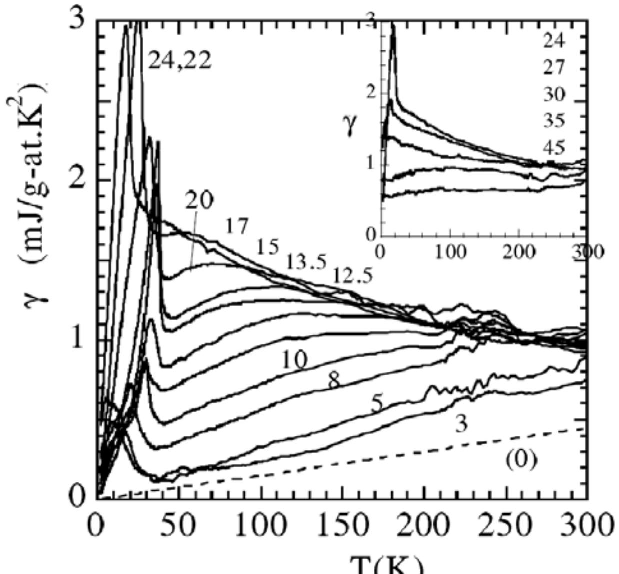


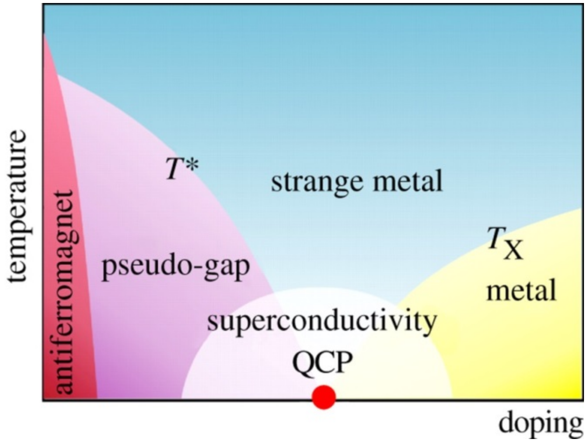
III. Phenomenology of the underdoped cuprates

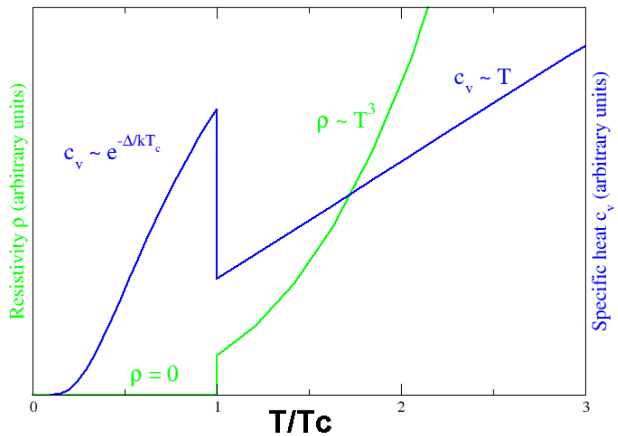
A. The pseudogap phenomenon in the normal state

- **Specific heat**
  - Linear T-dependence of specific heat coefficient γ above T_c
  - γ for $YBa_2Cu_3O_{6+y}$ for different y; optimally doped curves in the inset
  - γ for $La_{2-x}Sr_xCuO_4$ for different x; overdoped curves in the inset
  - γ at T_c reduces with decreasing doping









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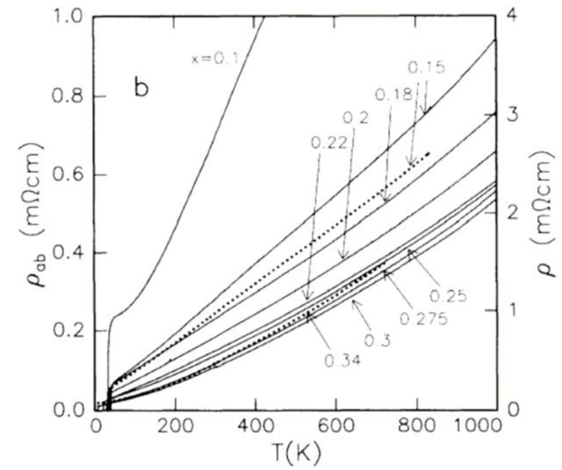
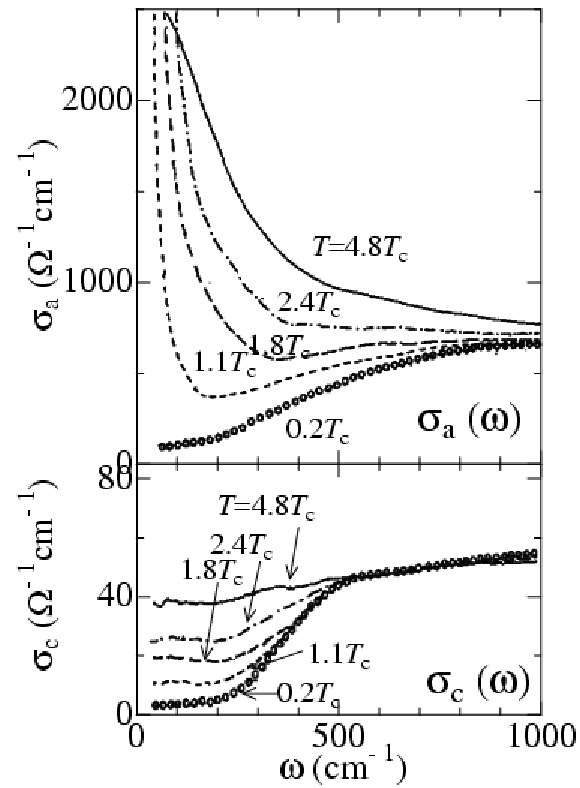
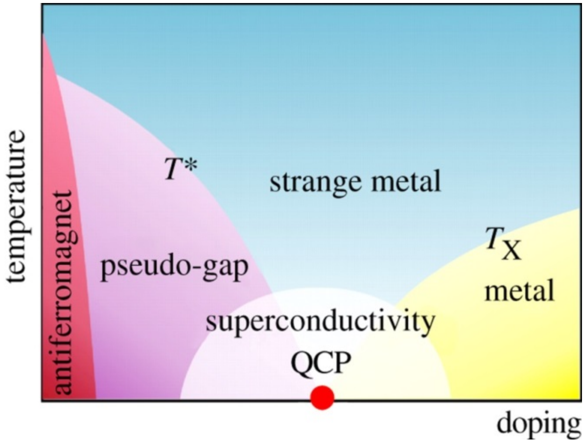
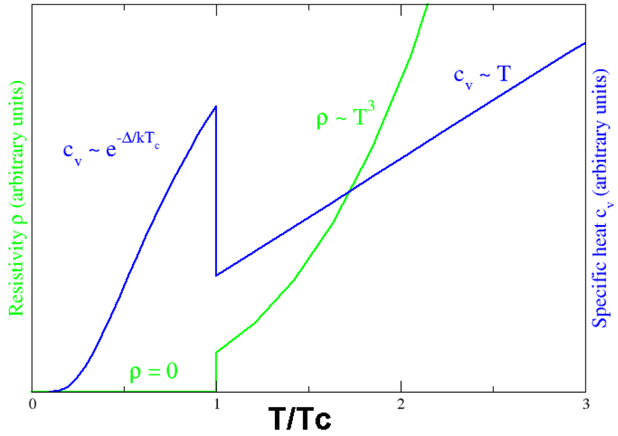
A. The pseudogap phenomenon in the normal state



- **DC Conductivity**
  - Anomalous linear-T "normal" state resistivity
- **AC Conductivity**
  - In-plane (CuO_2 plane) conductivity (σ_a) only gapped below T_c
  - Perpendicular conductivity (σ_c) gapped in the pseudogap phase







III. Phenomenology of the underdoped cuprates

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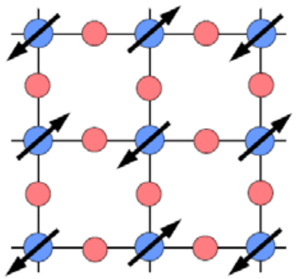
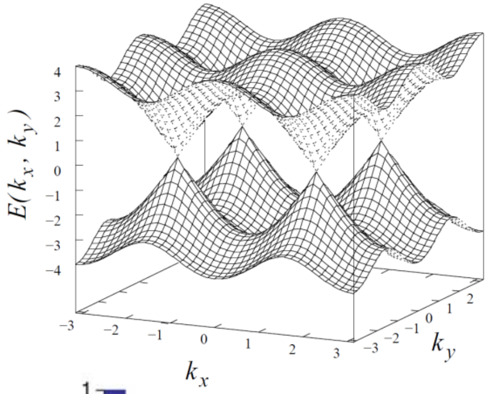
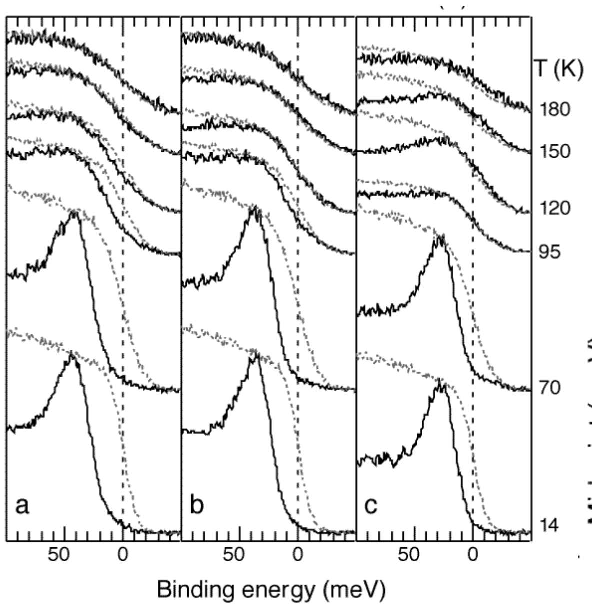
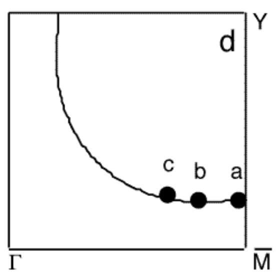
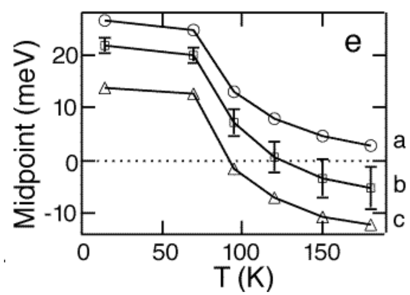


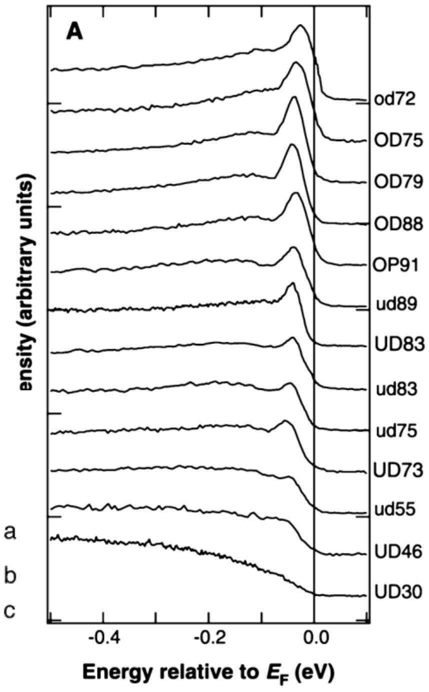
- **ARPES**
  - Superconducting gap exhibits nodes
  - Pseudogap opens at $(\pi/a, 0)$
  - Luttinger's theorem ➔ Fermi surface volume = $1 - x$
  - Spectral weight in coherence peak vanishes with decreasing hole doping

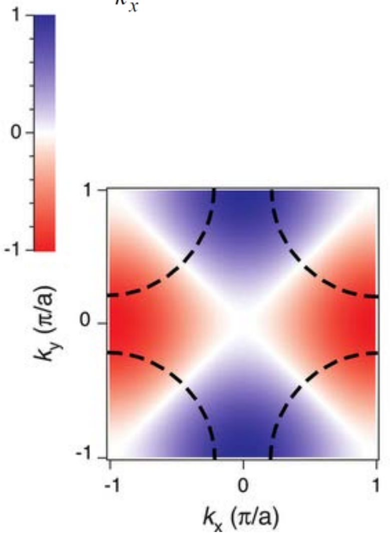








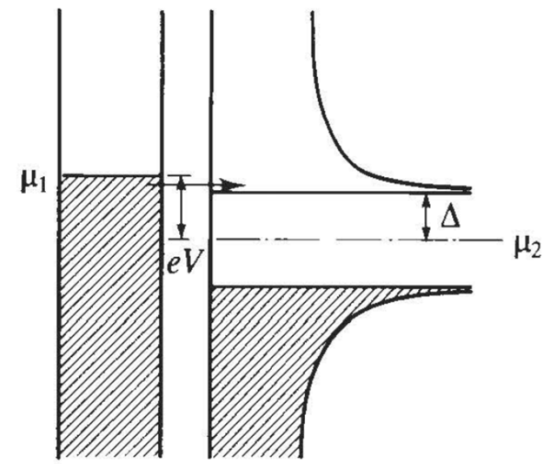


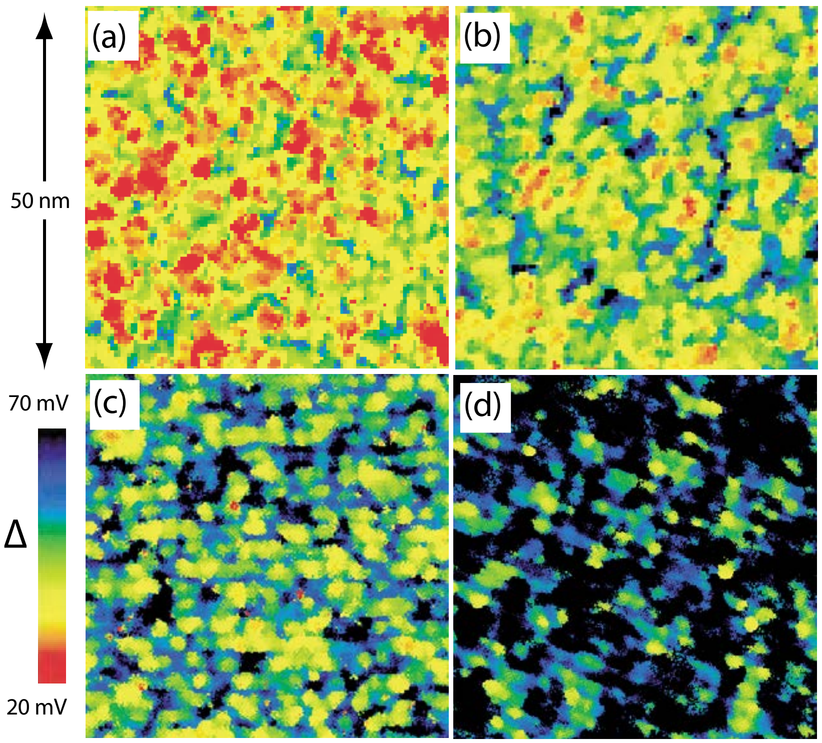


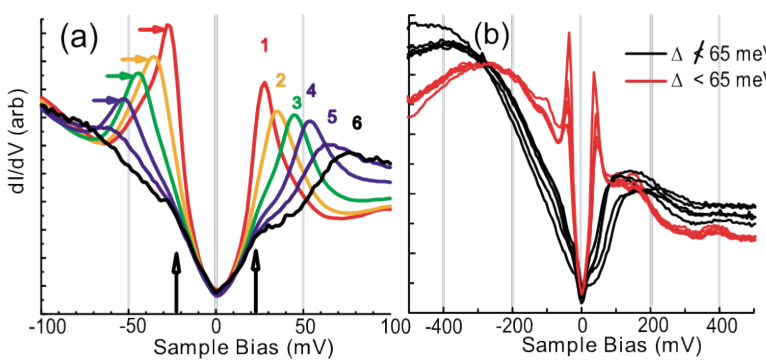
III. Phenomenology of the underdoped cuprates

A. The pseudogap phenomenon in the normal state

- **STM**
  - Surface inhomogeneity in the gap function
  - STM sees two dips ➔ first dip is indication of pseudogap state







(a)-(d) ➔ decreasing hole doping

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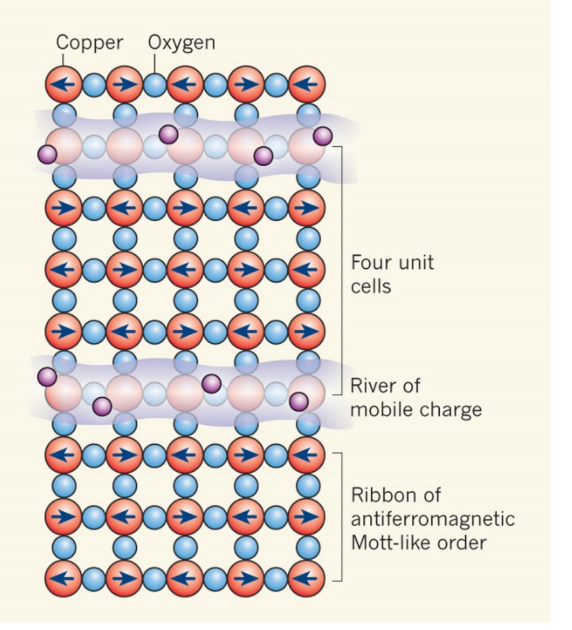
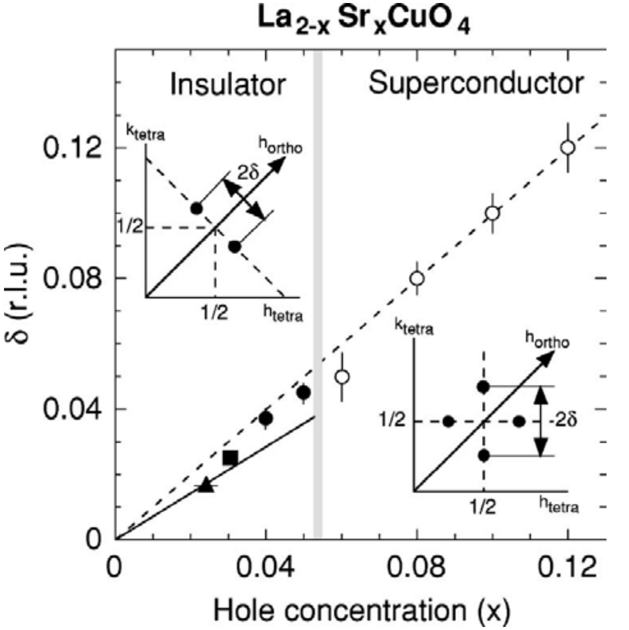
III. Phenomenology of the underdoped cuprates

B. Neutron scattering, resonance and stripes



- **Stripe order**
  - Observed in LSCO at doping of $x = 1/8$
  - Charge density wave (CDW) periodicity = 4
  - Spin density wave (SDW) periodicity = 8
- **Neutron scattering**
  - Scattering peak at $\mathbf{q} = (\pi/2, \pi/2)$
  - Incommensurability (δ) scales with doping (x)
  - "Fluctuating stripes" *apparently* invisible to experimental probes
  - Fluctuating stripes "may" explain pseudogap and superconductivity



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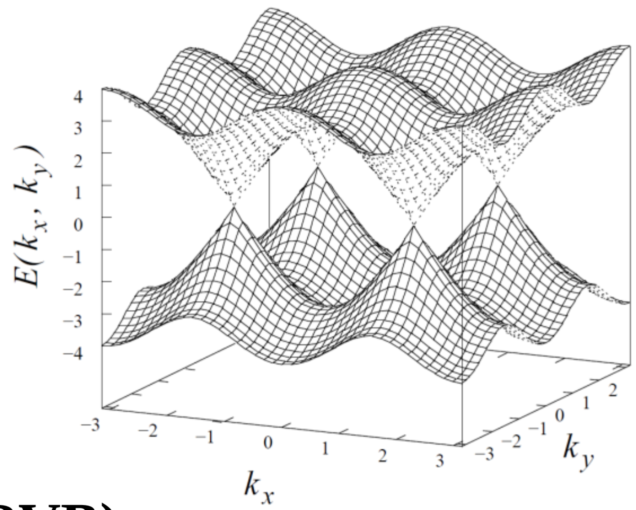

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III. Phenomenology of the underdoped cuprates

C. Quasiparticles in the superconducting state

- **Volovik effect**

  - Shift in quasiparticle energies

$$E_{\mathbf{A}}(\mathbf{k}) = E(\mathbf{k}) + \left(\frac{1}{2e}\nabla\theta - \mathbf{A}\right)\cdot \mathbf{j}_{\mathbf{k}}$$

  - Original quasiparticle spectrum

$$E(\mathbf{k}) = \left((\varepsilon_{\mathbf{k}} - \mu)^2 + \Delta_{\mathbf{k}}^2\right)^{1/2}$$

$$\Delta_{\mathbf{k}} = \frac{\Delta_0}{2}\left(\cos(k_x a) - \cos(k_y a)\right)$$

$$\varepsilon_{\mathbf{k}} = 2t\left(\cos(k_x a) + \cos(k_y a)\right)$$

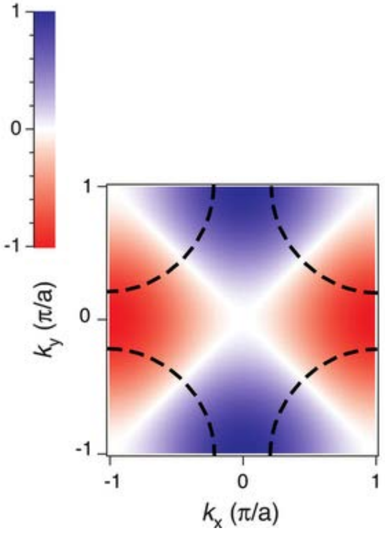
  - Nodal quasiparticle disperses like "normal" current

$$\mathbf{j}_{\mathbf{k}} = -e\frac{\partial \epsilon_{\mathbf{k}}}{\partial \mathbf{k}} \neq -e\frac{\partial E_{\mathbf{k}}}{\partial \mathbf{k}}$$

  - Phase winding around a vortex

$$|\nabla\theta(\mathbf{r})| \sim \frac{2\pi}{|\mathbf{r}|} \qquad \langle|\nabla\theta(\mathbf{r})|\rangle \sim \frac{\pi}{R} \qquad R \propto \left(\frac{\phi_0}{H}\right)^{1/2} \qquad \phi_0 = \frac{hc}{2e}$$

  - Field-dependent quasiparticle shift $\approx e v_F \left(H/\phi_0\right)^{1/2}$



III. Phenomenology of the underdoped cuprates

C. Quasiparticles in the superconducting state

- **Nodal quasiparticles**
  - Universal conductivity per layer

$$\frac{\kappa}{T} = \frac{k_B^2}{3\hbar c}\left(\frac{v_F}{v_\Delta} + \frac{v_\Delta}{v_F}\right)$$

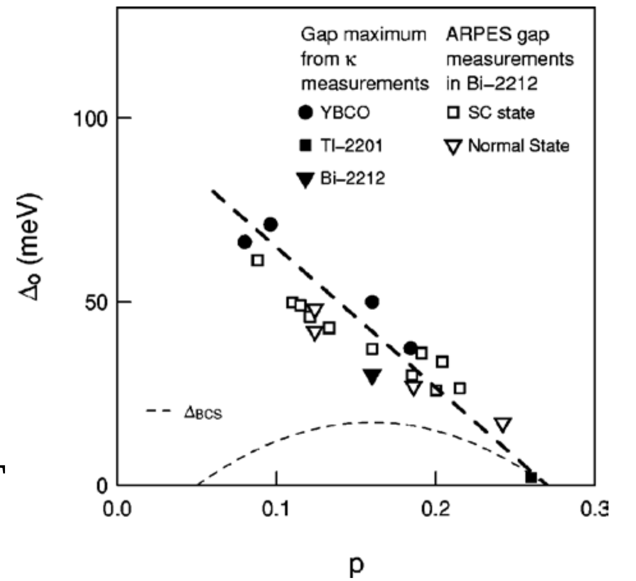
  - Antinodal gap obtained from extrapolation

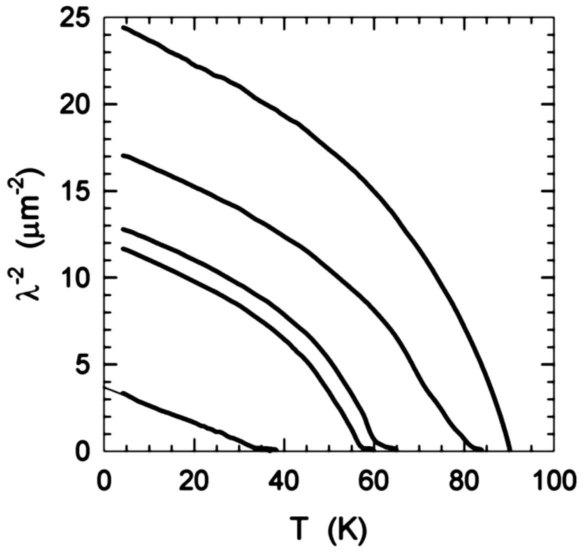
$$\Delta = \Delta_0 \cos(2\phi)$$

  - Phenomenological expression for linear-T superfluid density

$$\frac{n_s(T)}{m} = \frac{n_s(0)}{m} - \frac{2\ln(2)}{\pi}\alpha^2\left(\frac{v_F}{v_\Delta}\right)T$$

  - London penetration depth shows α = constant
  - Slave boson theory predicts $\alpha \propto x$





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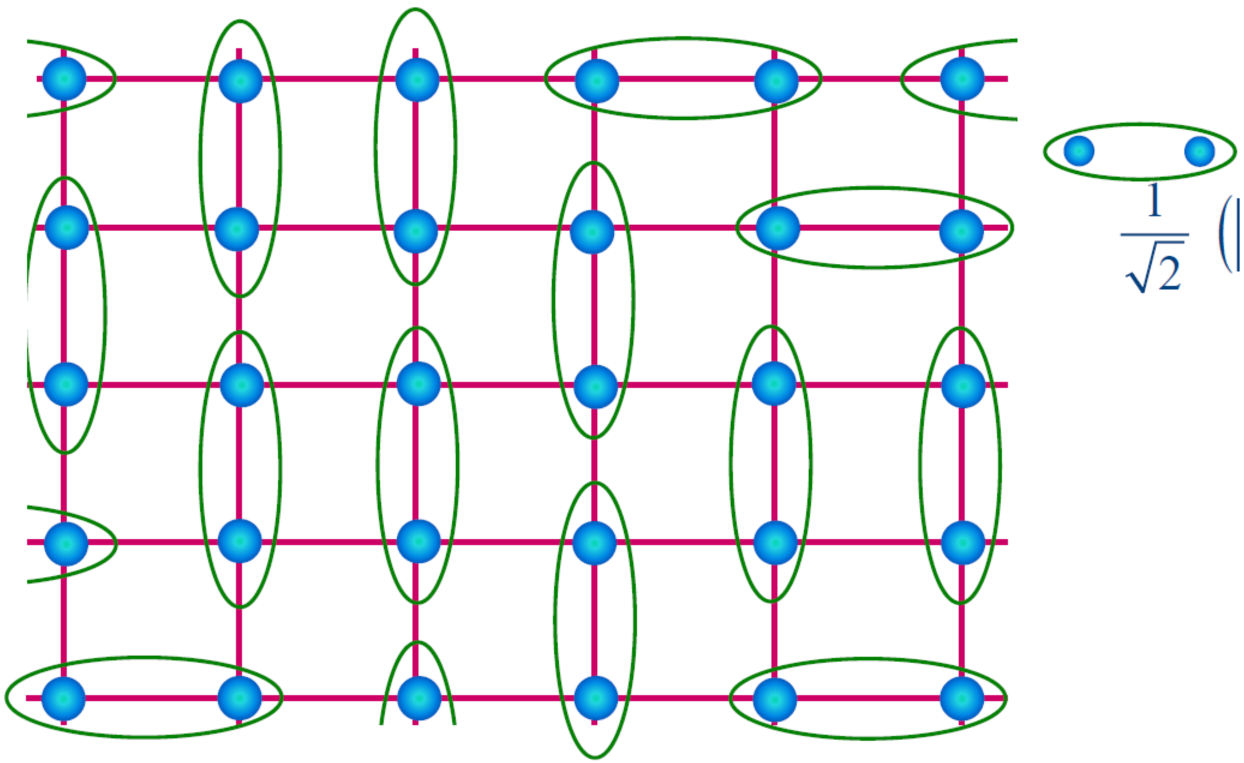
IV. Introduction to RVB and a simple explanation of the pseudogap

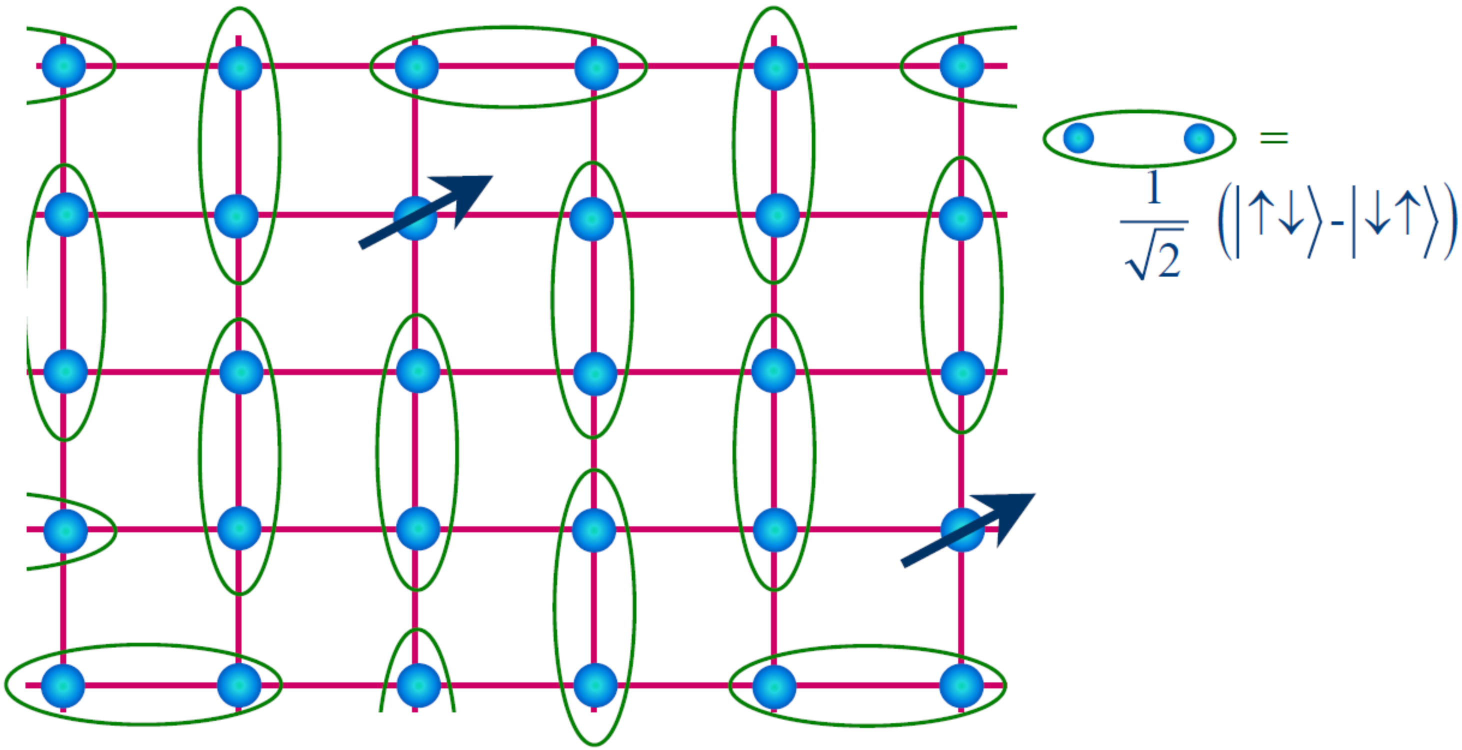
- **Resonating Valence Bond (RVB)**
  - Anderson revived RVB for the high-T_c problem
  - RVB state "soup" of fluctuating spin singlets



$$= \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

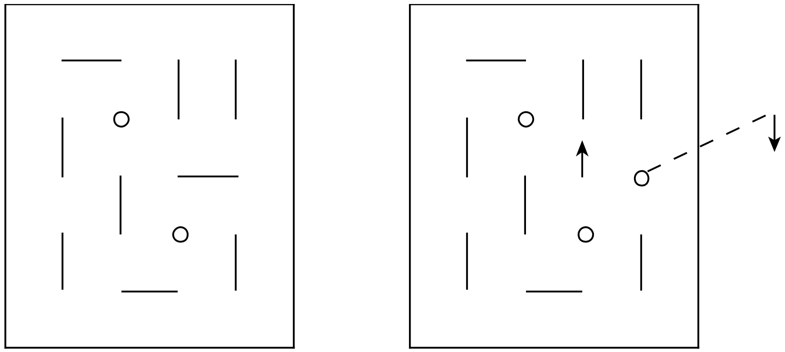
IV. Introduction to RVB and a simple explanation of the pseudogap

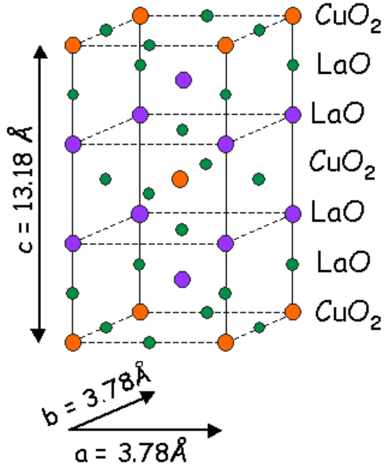
- **Deconfinement of “slave particle”**
  - We can “split” an electron into charge and spin degrees of freedom
  - Purely spin degrees of freedom ➔ “spinons”

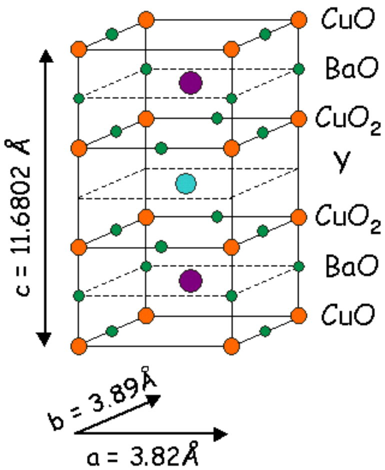


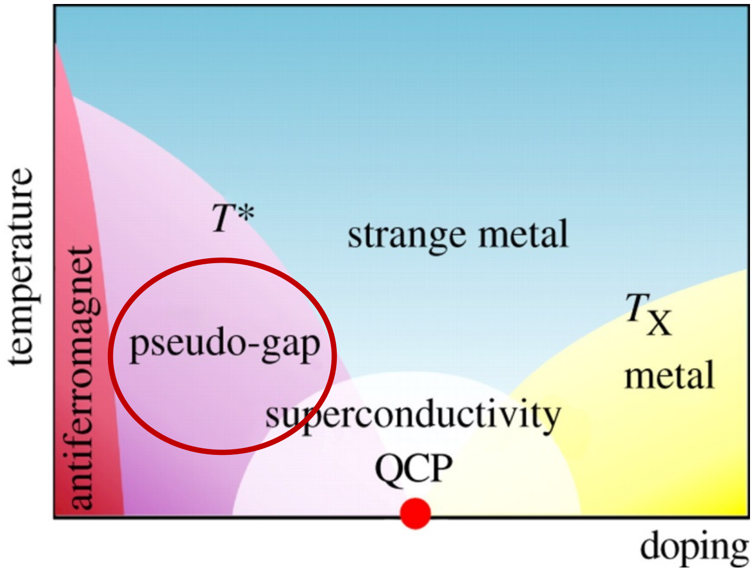
IV. Introduction to RVB and a simple explanation of the pseudogap

- **Resonating Valence Bond**
  - Anderson revived RVB for the high-T_c problem
  - Potential explanation of the pseudogap phase
  - Holes confined to 2D layers
  - Vertical motion of electrons needs breaking a singlet ➔ a gapped excitation









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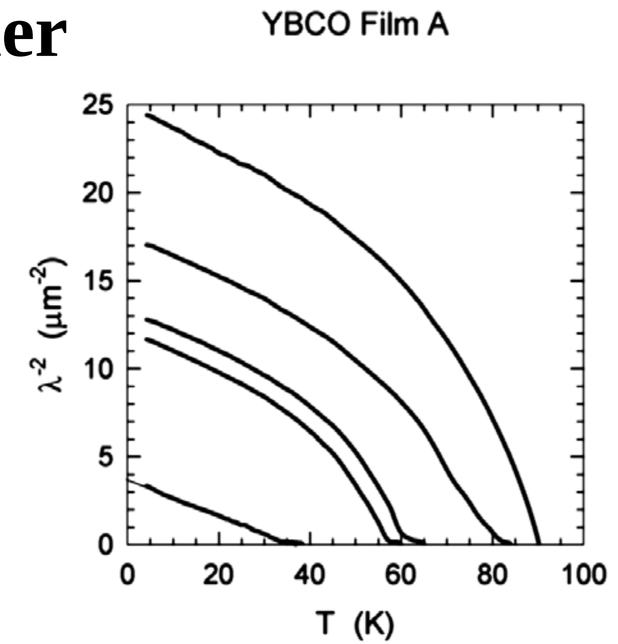
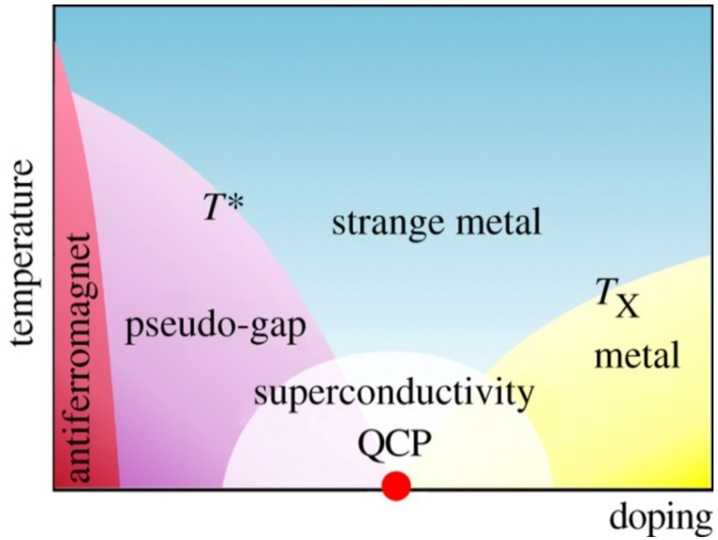
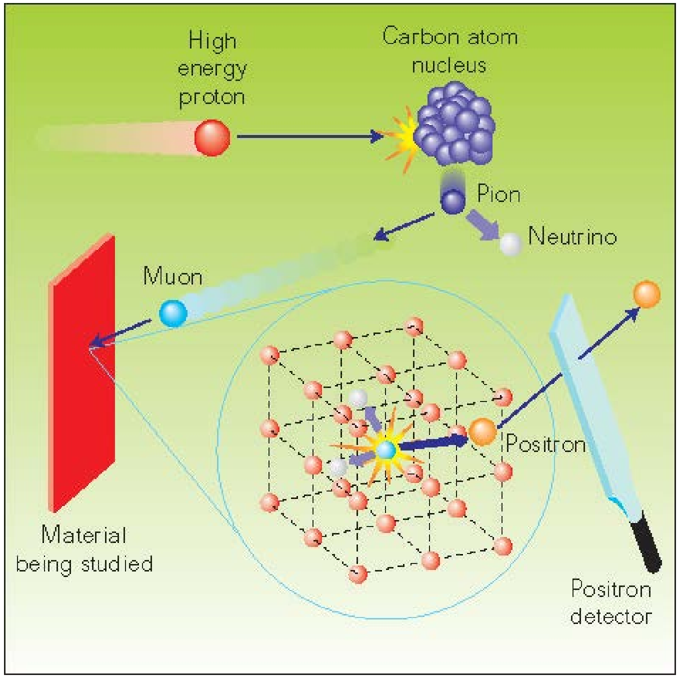
V. Phase fluctuation vs. competing order

- **Factors influencing T_c**
  - London penetration depth for field penetration perpendicular to the *ab* plane
$$\frac{1}{\lambda_\perp^2} = \frac{4\pi n_s^{3d} e^2}{m^* c^2} \propto xt$$
  - London penetration depth inferred from μSR rate $$\frac{1}{\lambda_\perp^2} \propto T_c$$
  - Indication of intralayer bose condensation of holes from μSR

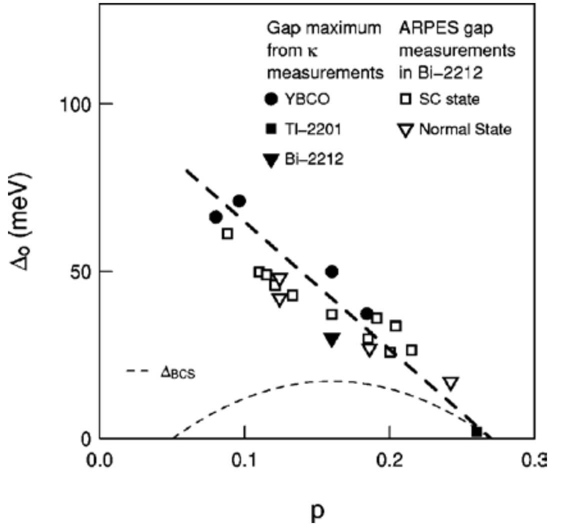






V. Phase fluctuation vs. competing order

A. Theory of T_c



- **T_c as a function of phase stiffness**
  - Phase stiffness of the order parameter $\Delta = |\Delta| e^{i\theta}$

$$H = \frac{1}{2} K_s^0 (\nabla\theta)^2 \qquad K_s = \frac{1}{4}\frac{\hbar^2 n_s}{m^*} = \frac{1}{4}\frac{\hbar^2 n_s^{3d} c_0}{m^*}$$

  - The BKT transition; energy of a single vortex

$$E_{\text{vortex}} = E_c + 2\pi K_s^0 \ln\left(\frac{L}{\xi_0}\right)$$

$$E_c = E_{\text{cond}}\xi_0^2 = \left(\frac{1}{2a^2}\mathcal{N}(0)\Delta_0^2\right)\left(\frac{v_F}{\Delta_0}\right)^2 = \frac{\mathcal{N}(0)}{m^* a^2}\left(\frac{1}{2}m^* v_F^2\right) = \frac{\mathcal{N}(0)}{m^* a^2} E_F$$

  - Relation between phase stiffness and T_c

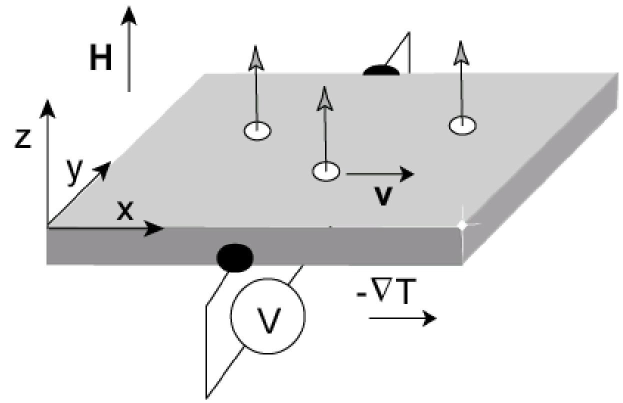
$$k_B T_c = \frac{\pi}{2} K_s(T_c) = \frac{\pi}{8}\frac{\hbar^2 n_s}{m^*}$$

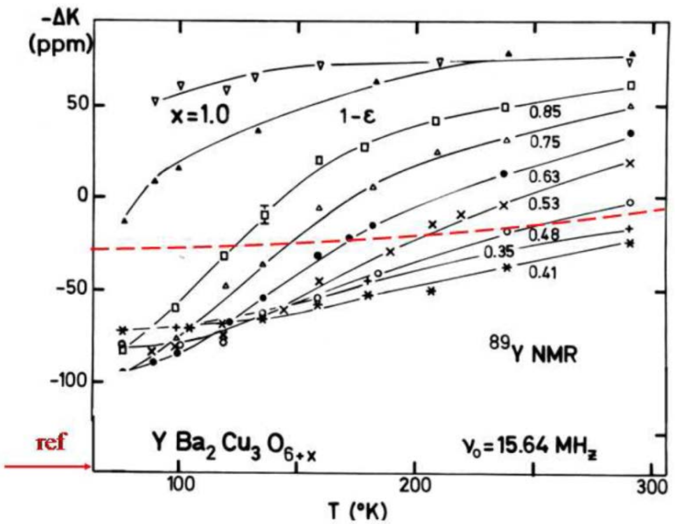
- **Cheap vortices**
  - Suppose T_{MF} is described by the standard BCS theory $E_c \approx \frac{\Delta_0^2}{E_F a^2}\xi_0^2 \approx E_F$
  - $E_F \approx E_c \gg k_B T_c$ ➔ Pseudogap *mostly* superconducting ➔ E_c is clearly not of order E_F
  - $E_c \approx T_c \approx K_s$ ➔ notion of strong phase fluctuations is applicable only on a temperature scale of $2T_c$

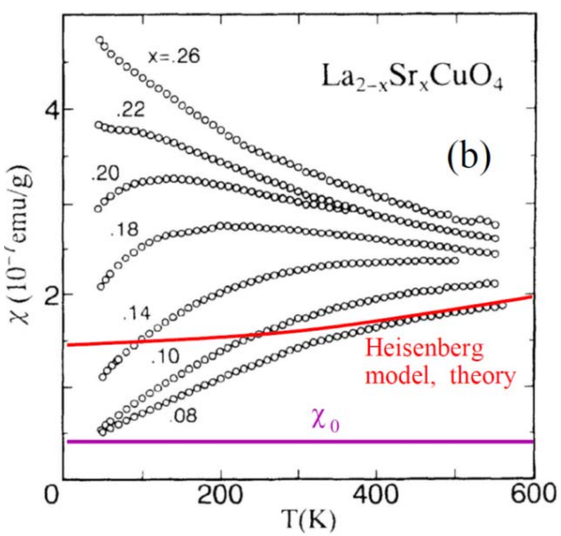
V. Phase fluctuation vs. competing order

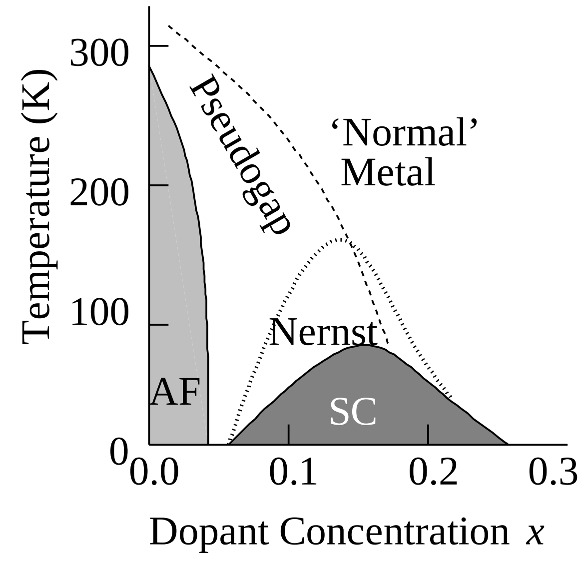
B. Cheap vortices and the Nernst effect

- **Nernst effect**
  - Transverse voltage due to longitudinal thermal gradient in the presence of a magnetic field
  - Nernst region as *second* type of pseudogap ➔ explained by phase fluctuations
  - The *first* type of pseudogap explained by singlet formation









Cuprates overview

- **Introduction and Phenomenology**
- **Experiments**
  - Pseudogap
  - Stripes
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VI. Projected trial wavefunctions and other numerical results

- Anderson's original RVB proposal

$$\Psi = P_G|\psi_0\rangle$$

- The Gutzwiller projection operator

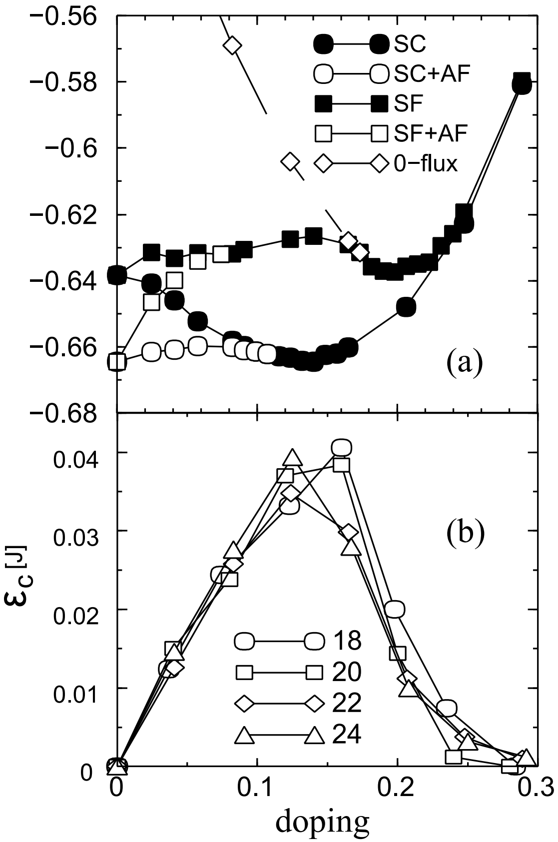
$$P_G = \prod_{\mathbf{i}} (1 - n_{\mathbf{i},\uparrow} n_{\mathbf{i},\downarrow})$$

- Projection operator too complicated to treat analytically

- Properties of the trial wave function evaluated using Monte Carlo sampling

- **Wave function ansatz**

SC: superconducting without antiferromagnetism
SC+AF: superconducting with antiferromagnetism
SF: staggered-flux without antiferromagnetism
SF+AF: staggered-flux with antiferromagnetism
ZF: zero-flux



VI. Projected trial wavefunctions and other numerical results

A. The half-filled case

- *d*-wave BCS trial wavefunction

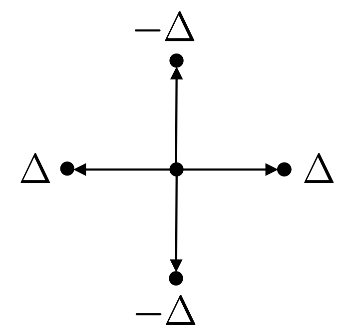
$$H_d = -\sum_{\langle \mathbf{ij}\rangle,\sigma}\left(\chi_{\mathbf{ij}} f^\dagger_{\mathbf{i},\sigma} f_{\mathbf{j},\sigma} + \text{c.c.}\right) - \sum_{\mathbf{i},\sigma} \mu f^\dagger_{\mathbf{i},\sigma} f_{\mathbf{i},\sigma} + \sum_{\langle \mathbf{ij}\rangle}\left[\Delta_{\mathbf{ij}}\left(f^\dagger_{\mathbf{i},\uparrow} f^\dagger_{\mathbf{j},\downarrow} - f^\dagger_{\mathbf{i},\downarrow} f^\dagger_{\mathbf{j},\uparrow}\right) + \text{c.c.}\right]$$

$$\chi_{\mathbf{ij}} = \chi_0 \qquad \Delta_{\mathbf{ij}} = \Delta_0 \quad \text{for} \quad \mathbf{j} = \mathbf{i} + \hat{\mathbf{x}} \qquad \Delta_{\mathbf{ij}} = -\Delta_0 \quad \text{for} \quad \mathbf{j} = \mathbf{i} + \hat{\mathbf{j}}$$

$$E_{\mathbf{k}} = \sqrt{(\epsilon_{\mathbf{k}} - \mu)^2 + \Delta^2_{\mathbf{k}}} \qquad |\psi_0\rangle = \prod_{\mathbf{k}}\left(u_{\mathbf{k}} + v_{\mathbf{k}} f^\dagger_{\mathbf{k}\uparrow} f^\dagger_{-\mathbf{k}\downarrow}\right)|0\rangle$$

$$\epsilon_{\mathbf{k}} = -2\chi_0\left(\cos(k_x) + \cos(k_y)\right) \qquad |v_{\mathbf{k}}|^2 = 1 - |u_{\mathbf{k}}|^2 = \frac{1}{2}\left(1 - \frac{\epsilon_{\mathbf{k}}}{E_{\mathbf{k}}}\right)^{1/2}$$

$$\Delta_{\mathbf{k}} = 2\Delta_0\left(\cos(k_x) - \cos(k_y)\right)$$

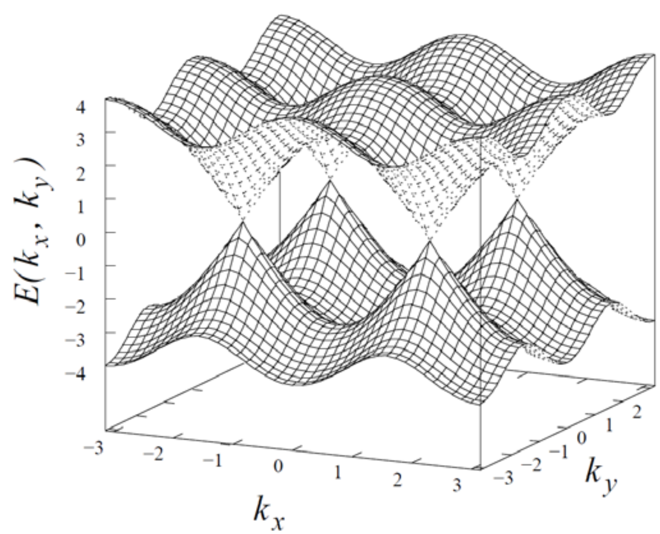


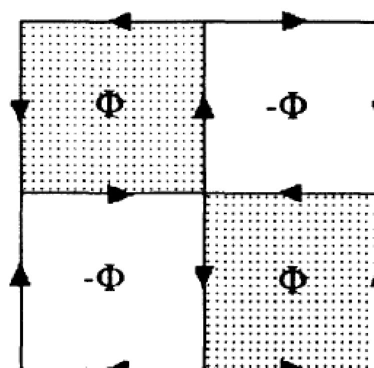
d-wave

- Staggered flux state

$$\chi_{\mathbf{ij}} = \chi_0 \exp\left(i(-1)^{i_x + j_y}\Phi_0\right)$$

$$\tan(\Phi_0) = \frac{\Delta_0}{\chi_0}$$





- SU(2) symmetry

$$f^\dagger_{\mathbf{i},\uparrow} \to \alpha_{\mathbf{i}} f^\dagger_{\mathbf{i},\uparrow} + \beta_{\mathbf{i}} f_{\mathbf{i},\downarrow}$$

$$f_{\mathbf{i},\downarrow} \to -\beta^*_{\mathbf{i}} f^\dagger_{\mathbf{i},\uparrow} + \alpha^*_{\mathbf{i}} f_{\mathbf{i},\downarrow}$$

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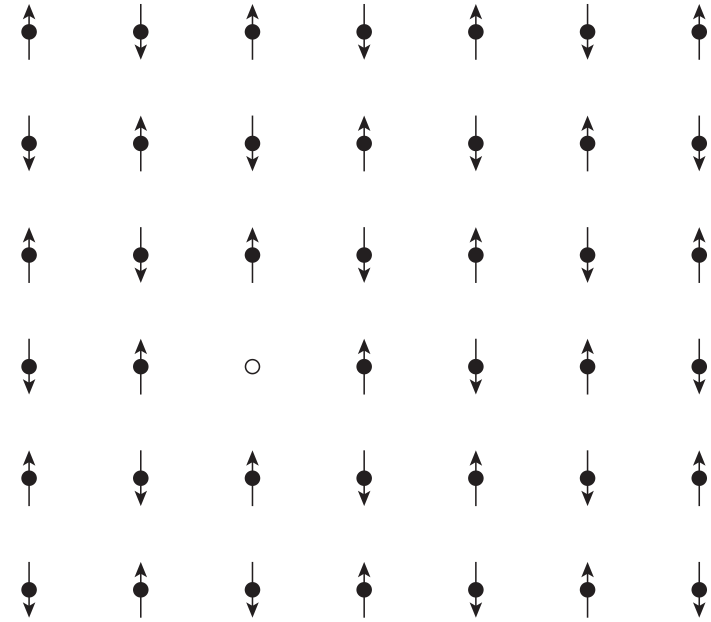
VII. The single hole problem



- **Vacancy in an "antiferromagnetic sea"**
  - Dynamics of a single hole

$$H = \frac{t}{N}\sum_{\mathbf{k},q} M_{\mathbf{k},\mathbf{q}}\left[h^\dagger_{\mathbf{k}} h_{\mathbf{k}-\mathbf{q}}\alpha_{\mathbf{q}} + \text{h.c.}\right] + \sum_q \Omega_{\mathbf{q}}\alpha^\dagger_{\mathbf{q}}\alpha_{\mathbf{q}}$$

$$\Omega_{\mathbf{q}} = 2J\sqrt{1-\gamma^2_{\mathbf{q}}} \qquad \gamma_{\mathbf{q}} = \frac{1}{2}\left[\cos(q_x)+\cos(q_y)\right]$$

$$M(\mathbf{k},\mathbf{q}) = 4(u_{\mathbf{q}}\gamma_{\mathbf{k}-\mathbf{q}} + v_{\mathbf{q}}\gamma_{\mathbf{k}})$$

$$u_{\mathbf{k}} = \sqrt{\frac{1+\nu_{\mathbf{k}}}{2\nu_{\mathbf{k}}}} \qquad v_{\mathbf{k}} = -\text{sign}(\gamma_{\mathbf{k}})\sqrt{\frac{1-\nu_{\mathbf{k}}}{2\nu_{\mathbf{k}}}} \qquad \nu_{\mathbf{k}} = \sqrt{1-\gamma^2_{\mathbf{k}}}$$

  - Using self-consistent Born approximation, and ignoring crossing magnon propagators, self-consistent equation for the hole propagator is

$$G(\mathbf{k},\omega) = \left[\omega - \sum_{\mathbf{q}} g^2(\mathbf{k},\mathbf{q})G(\mathbf{k}-\mathbf{q},\omega-\Omega_{\mathbf{q}})\right]^{-1} \qquad A(\mathbf{k},\omega) = -(1/\pi)\Im G^R(\mathbf{k},\omega)$$

  - ARPES sees two peaks in $A(\mathbf{k}, \omega)$ in addition to hole quasiparticle peaks centered at

$$E_n/t = -b + a_n\left(\frac{J}{t}\right)^{2/3}$$

  - These can be understood as the "string" excitation of the hole moving in the linear confining potential due to the AF background

VII. The single hole problem

- **Vacancy in an "antiferromagnetic sea"**
  - ARPES sees two peaks in $A(\mathbf{k}, \omega)$ in addition to hole quasiparticle peaks centered at

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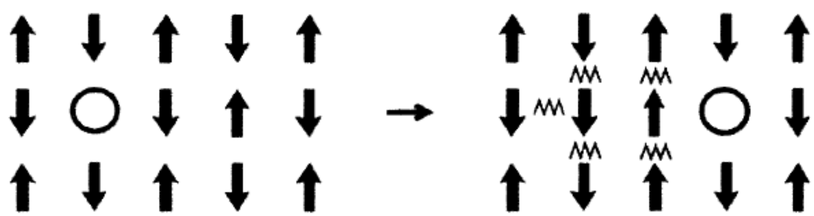
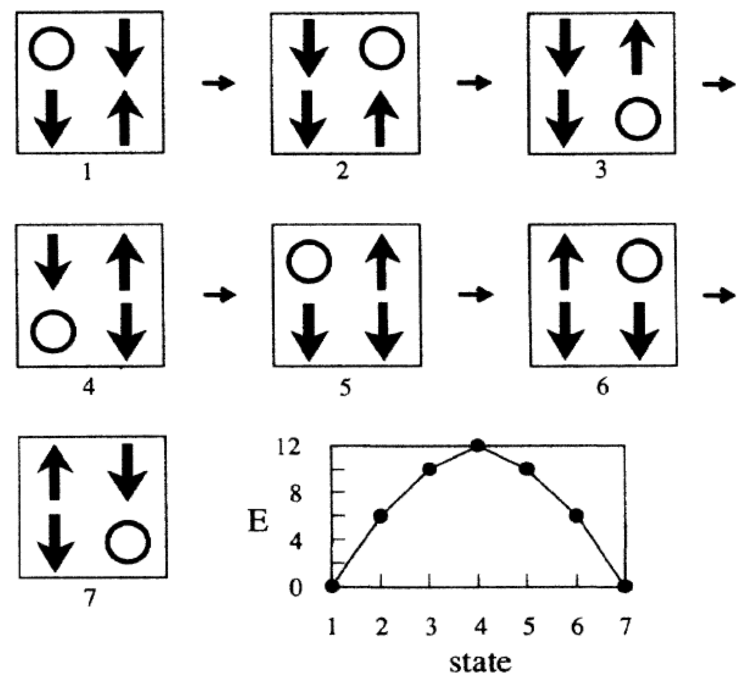
  - These can be understood as the "string" excitation of the hole moving in the linear confining potential due to the AF background



  - The hole must retrace its path to "kill" the string ➔ holes are localized
  - Do holes really conduct?
  - Yes! A hole does **not** *necessarily* need to retrace its path without raising energy



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VIII. Slave boson formulation of t-J model and mean field theory

- **Splitting the electron**

  - Low energy physics in terms of the *t-J* model

$$H = \sum_{\langle \mathbf{ij} \rangle} J \left(\mathbf{S_i} \cdot \mathbf{S_j} - \frac{1}{4} n_\mathbf{i} n_\mathbf{j} \right) - \sum_{\mathbf{ij}} t_\mathbf{ij} \left(c^\dagger_{\mathbf{i}\sigma} c_{\mathbf{j}\sigma} + \text{H.c.} \right)$$

  - No-double-occupancy condition

$$\sum_\sigma c^\dagger_{\mathbf{i}\sigma} c_{\mathbf{i}\sigma} \leq 1$$

  - Most general "electron splitting" using slave boson operators

$$c^\dagger_{\mathbf{i}\sigma} = f^\dagger_{\mathbf{i}\sigma} b_\mathbf{i} + \epsilon_{\sigma\sigma'} f_{\mathbf{i}\sigma'} d^\dagger_\mathbf{i} \qquad f^\dagger_{\mathbf{i}\uparrow} f_{\mathbf{i}\uparrow} + f^\dagger_{\mathbf{i}\downarrow} f_{\mathbf{i}\downarrow} + b^\dagger_\mathbf{i} b_\mathbf{i} + d^\dagger_\mathbf{i} d_\mathbf{i} = 1$$

  - Enforcing no-double-occupancy condition in terms of slave particles

$$c^\dagger_{\mathbf{i}\sigma} = f^\dagger_{\mathbf{i}\sigma} b_\mathbf{i} \qquad f^\dagger_{\mathbf{i}\uparrow} f_{\mathbf{i}\uparrow} + f^\dagger_{\mathbf{i}\downarrow} f_{\mathbf{i}\downarrow} + b^\dagger_\mathbf{i} b_\mathbf{i} = 1$$

  - Heisenberg exchange term in terms of slave particles

$$\mathbf{S_i} \cdot \mathbf{S_j} = -\frac{1}{4} f^\dagger_{\mathbf{i}\sigma} f_{\mathbf{j}\sigma} f^\dagger_{\mathbf{j}\beta} f_{\mathbf{i}\beta} - \frac{1}{4} \left(f^\dagger_{\mathbf{i}\uparrow} f^\dagger_{\mathbf{j}\downarrow} - f^\dagger_{\mathbf{i}\downarrow} f^\dagger_{\mathbf{j}\uparrow} \right) \left(f_{\mathbf{j}\downarrow} f_{\mathbf{i}\uparrow} - f_{\mathbf{j}\uparrow} f_{\mathbf{i}\downarrow} \right) + \frac{1}{4} \left(f^\dagger_{\mathbf{i}\alpha} f_{\mathbf{i}\alpha} \right)$$

$$n_\mathbf{i} n_\mathbf{j} = (1 - b^\dagger_\mathbf{i} b_\mathbf{i})(1 - b^\dagger_\mathbf{j} b_\mathbf{j})$$

VIII. Slave boson formulation of t-J model and mean field theory

- **Splitting the electron**
  - Heisenberg exchange term in terms of slave particles

$$\mathbf{S_i}\cdot\mathbf{S_j} = -\frac{1}{4}f^\dagger_{\mathbf{i}\sigma}f_{\mathbf{j}\sigma}f^\dagger_{\mathbf{j}\beta}f_{\mathbf{i}\beta} - \frac{1}{4}\left(f^\dagger_{\mathbf{i}\uparrow}f^\dagger_{\mathbf{j}\downarrow} - f^\dagger_{\mathbf{i}\downarrow}f^\dagger_{\mathbf{j}\uparrow}\right)\left(f_{\mathbf{j}\downarrow}f_{\mathbf{i}\uparrow} - f_{\mathbf{j}\uparrow}f_{\mathbf{i}\downarrow}\right) + \frac{1}{4}\left(f^\dagger_{\mathbf{i}\alpha}f_{\mathbf{i}\alpha}\right)$$

$$n_\mathbf{i}n_\mathbf{j} = (1 - b^\dagger_\mathbf{i}b_\mathbf{i})(1 - b^\dagger_\mathbf{j}b_\mathbf{j}) \qquad f^\dagger_{\mathbf{i}\uparrow}f_{\mathbf{i}\uparrow} + f^\dagger_{\mathbf{i}\downarrow}f_{\mathbf{i}\downarrow} + b^\dagger_\mathbf{i}b_\mathbf{i} = 1$$

  - Decoupling exchange term in particle-hole and particle-particle channels

$\mathbf{S_i}\cdot\mathbf{S_j} - \frac{1}{4}n_\mathbf{i}n_\mathbf{j}$ evaluated using constraint and ignoring $\frac{1}{4}b^\dagger_\mathbf{i}b_\mathbf{i}b^\dagger_\mathbf{j}b_\mathbf{j}$

  - Hubbard-Stratonovich transformation

$$Z = \int Df\,Df^\dagger\,Db\,D\lambda\,D\chi\,D\Delta\,\exp\left(-\int_0^\beta d\tau L_1\right)$$

$$L_1 = \tilde{J}\sum_{\langle\mathbf{ij}\rangle}\left(|\chi_\mathbf{ij}|^2 + |\Delta_\mathbf{ij}|^2\right) + \sum_{\mathbf{i}\sigma}f^\dagger_{\mathbf{i}\sigma}(\partial_\tau - i\lambda_\mathbf{i})f_{\mathbf{i}\sigma} - \tilde{J}\left[\sum_{\langle\mathbf{ij}\rangle}\chi^*_\mathbf{ij}\left(\sum_\sigma f^\dagger_{\mathbf{i}\sigma}f_{\mathbf{j}\sigma}\right) + \text{c.c.}\right]$$

$$+\tilde{J}\left[\sum_{\langle\mathbf{ij}\rangle}\Delta_\mathbf{ij}\left(f^\dagger_{\mathbf{i}\uparrow}f^\dagger_{\mathbf{j}\downarrow} - f^\dagger_{\mathbf{i}\downarrow}f^\dagger_{\mathbf{j}\uparrow}\right) + \text{c.c.}\right] + \sum_\mathbf{i}b^*_\mathbf{i}(\partial_\tau - i\lambda_\mathbf{i} + \mu_B)b_\mathbf{i} - \sum_\mathbf{ij}t_\mathbf{ij}b_\mathbf{i}b^*_\mathbf{j}f^\dagger_{\mathbf{i}\sigma}f_{\mathbf{j}\sigma}$$

VIII. Slave boson formulation of t-J model and mean field theory

- **"Local" U(1) gauge symmetry**
  - Effective Lagrangian

$$L_1 = \tilde{J} \sum_{\langle \mathbf{ij} \rangle} \left(|\chi_{\mathbf{ij}}|^2 + |\Delta_{\mathbf{ij}}|^2 \right) + \sum_{\mathbf{i}\sigma} f^\dagger_{\mathbf{i}\sigma} (\partial_\tau - i\lambda_{\mathbf{i}}) f_{\mathbf{i}\sigma} - \tilde{J} \left[\sum_{\langle \mathbf{ij} \rangle} \chi^*_{\mathbf{ij}} \left(\sum_\sigma f^\dagger_{\mathbf{i}\sigma} f_{\mathbf{j}\sigma} \right) + \text{c.c.} \right]$$

$$+ \tilde{J} \left[\sum_{\langle \mathbf{ij} \rangle} \Delta_{\mathbf{ij}} \left(f^\dagger_{\mathbf{i}\uparrow} f^\dagger_{\mathbf{j}\downarrow} - f^\dagger_{\mathbf{i}\downarrow} f^\dagger_{\mathbf{j}\uparrow} \right) + \text{c.c.} \right] + \sum_{\mathbf{i}} b^*_{\mathbf{i}} (\partial_\tau - i\lambda_{\mathbf{i}} + \mu_B) b_{\mathbf{i}} - \sum_{\mathbf{ij}} t_{\mathbf{ij}} b_{\mathbf{i}} b^*_{\mathbf{j}} f^\dagger_{\mathbf{i}\sigma} f_{\mathbf{j}\sigma}$$

  - Local U(1) transformation

$$f_{\mathbf{i}} \to e^{i\theta_{\mathbf{i}}} f_{\mathbf{i}} \qquad b_{\mathbf{i}} \to e^{i\theta_{\mathbf{i}}} b_{\mathbf{i}} \qquad \chi_{\mathbf{ij}} \to e^{-i\theta_{\mathbf{i}}} \chi_{\mathbf{ij}} e^{i\theta_{\mathbf{j}}} \qquad \Delta_{\mathbf{ij}} \to e^{i\theta_{\mathbf{i}}} \Delta_{\mathbf{ij}} e^{i\theta_{\mathbf{j}}} \qquad \lambda_{\mathbf{i}} \to \lambda_{\mathbf{i}} + \partial_\tau \theta_{\mathbf{i}}$$

  - Phase fluctuations of $\chi_{\mathbf{ij}}$ and $\lambda_{\mathbf{i}}$ have dynamics of U(1) gauge field
  - We have various choices satisfying mean field conditions

$$\chi_{\mathbf{ij}} = \sum_\sigma \langle f^\dagger_{\mathbf{i}\sigma} f_{\mathbf{j}\sigma} \rangle \qquad \Delta_{\mathbf{ij}} = \langle f_{\mathbf{i}\uparrow} f_{\mathbf{j}\downarrow} - f_{\mathbf{i}\downarrow} f_{\mathbf{i}\uparrow} \rangle$$

VIII. Slave boson formulation of t-J model and mean field theory

- **Mean field ansatz**
  - Effective Lagrangian

$$L_1 = \tilde{J}\sum_{\langle \mathbf{ij}\rangle}\left(|\chi_{\mathbf{ij}}|^2 + |\Delta_{\mathbf{ij}}|^2\right) + \sum_{\mathbf{i}\sigma} f^\dagger_{\mathbf{i}\sigma}(\partial_\tau - i\lambda_{\mathbf{i}})f_{\mathbf{i}\sigma} - \tilde{J}\left[\sum_{\langle \mathbf{ij}\rangle}\chi^*_{\mathbf{ij}}\left(\sum_\sigma f^\dagger_{\mathbf{i}\sigma}f_{\mathbf{j}\sigma}\right) + \text{c.c.}\right]$$

$$+\tilde{J}\left[\sum_{\langle \mathbf{ij}\rangle}\Delta_{\mathbf{ij}}\left(f^\dagger_{\mathbf{i}\uparrow}f^\dagger_{\mathbf{j}\downarrow} - f^\dagger_{\mathbf{i}\downarrow}f^\dagger_{\mathbf{j}\uparrow}\right) + \text{c.c.}\right] + \sum_{\mathbf{i}} b^*_{\mathbf{i}}(\partial_\tau - i\lambda_{\mathbf{i}} + \mu_B)b_{\mathbf{i}} - \sum_{\mathbf{ij}} t_{\mathbf{ij}}b_{\mathbf{i}}b^*_{\mathbf{j}}f^\dagger_{\mathbf{i}\sigma}f_{\mathbf{j}\sigma}$$

  - The uniform RVB (uRVB) state ➔ purely fermionic theory

$$H_{\text{uRVB}} = -\sum_{\mathbf{k}\sigma} 2\tilde{J}\chi\left(\cos(k_x) + \cos(k_y)\right) f^\dagger_{\mathbf{k}\sigma}f_{\mathbf{k}\sigma}$$

  - Lower energy states than uRVB state
    - *d*-wave state
    - Staggered flux state

  - *d*-wave and staggered flux state have identical dispersion due to local SU(2) symmetry

$$\Phi_{\mathbf{i}\uparrow} = \begin{pmatrix} f_{\mathbf{i}\uparrow} \\ f^\dagger_{\mathbf{i}\downarrow}\end{pmatrix} \quad \Phi_{\mathbf{i}\downarrow} = \begin{pmatrix} f_{\mathbf{i}\downarrow} \\ -f^\dagger_{\mathbf{i}\uparrow}\end{pmatrix}$$

VIII. Slave boson formulation of t-J model and mean field theory

- **Mean field ansatz**
  - Effective Lagrangian

$$L_1 = \tilde{J} \sum_{\langle \mathbf{ij} \rangle} \left(|\chi_{\mathbf{ij}}|^2 + |\Delta_{\mathbf{ij}}|^2 \right) + \sum_{\mathbf{i}\sigma} f^\dagger_{\mathbf{i}\sigma} (\partial_\tau - i\lambda_{\mathbf{i}}) f_{\mathbf{i}\sigma} - \tilde{J} \left[\sum_{\langle \mathbf{ij} \rangle} \chi^*_{\mathbf{ij}} \left(\sum_\sigma f^\dagger_{\mathbf{i}\sigma} f_{\mathbf{j}\sigma} \right) + \text{c.c.} \right]$$

$$+ \tilde{J} \left[\sum_{\langle \mathbf{ij} \rangle} \Delta_{\mathbf{ij}} \left(f^\dagger_{\mathbf{i}\uparrow} f^\dagger_{\mathbf{j}\downarrow} - f^\dagger_{\mathbf{i}\downarrow} f^\dagger_{\mathbf{j}\uparrow} \right) + \text{c.c.} \right] + \sum_{\mathbf{i}} b^*_{\mathbf{i}} (\partial_\tau - i\lambda_{\mathbf{i}} + \mu_B) b_{\mathbf{i}} - \sum_{\mathbf{ij}} t_{\mathbf{ij}} b_{\mathbf{i}} b^*_{\mathbf{j}} f^\dagger_{\mathbf{i}\sigma} f_{\mathbf{j}\sigma}$$

  - Use of SU(2) doublets

$$\Phi_{\mathbf{i}\uparrow} = \begin{pmatrix} f_{\mathbf{i}\uparrow} \\ f^\dagger_{\mathbf{i}\downarrow} \end{pmatrix} \quad \Phi_{\mathbf{i}\downarrow} = \begin{pmatrix} f_{\mathbf{i}\downarrow} \\ -f^\dagger_{\mathbf{i}\uparrow} \end{pmatrix}$$

  - Compact Effective Lagrangian

$$L_1 = \frac{\tilde{J}}{2} \sum_{\langle \mathbf{ij} \rangle} \text{Tr} \left[U^\dagger_{\mathbf{ij}} U_{\mathbf{ij}} \right] + \frac{\tilde{J}}{2} \sum_{\langle \mathbf{ij} \rangle, \sigma} \left(\Phi^\dagger_{\mathbf{i}\sigma} U_{\mathbf{ij}} \Phi_{\mathbf{j}\sigma} + \text{c.c.} \right) + \sum_{\mathbf{i}\sigma} f^\dagger_{\mathbf{i}\sigma} \left(\partial_\tau - i\lambda_{\mathbf{i}} \right) f_{\mathbf{i}\sigma}$$

$$+ \sum_{\mathbf{i}} b^*_{\mathbf{i}} \left(\partial_\tau - i\lambda_{\mathbf{i}} + \mu_B \right) b_{\mathbf{i}} - \sum_{\mathbf{ij}} t_{\mathbf{ij}} b_{\mathbf{i}} b^*_{\mathbf{j}} f^\dagger_{\mathbf{i}\sigma} f_{\mathbf{j}\sigma}$$

$$U_{\mathbf{ij}} = \begin{pmatrix} -\chi^*_{\mathbf{ij}} & \Delta_{\mathbf{ij}} \\ \Delta^*_{\mathbf{ij}} & \chi_{\mathbf{ij}} \end{pmatrix} \qquad \chi_{\mathbf{ij}} = \sum_\sigma \langle f^\dagger_{\mathbf{i}\sigma} f_{\mathbf{j}\sigma} \rangle \qquad \Delta_{\mathbf{ij}} = \langle f_{\mathbf{i}\uparrow} f_{\mathbf{j}\downarrow} - f_{\mathbf{i}\downarrow} f_{\mathbf{i}\uparrow} \rangle$$

VIII. Slave boson formulation of t-J model and mean field theory

- **Mean field ansatz**

$$\Phi_{\mathbf{i}\uparrow} = \begin{pmatrix} f_{\mathbf{i}\uparrow} \\ f_{\mathbf{i}\downarrow}^\dagger \end{pmatrix} \quad \Phi_{\mathbf{i}\downarrow} = \begin{pmatrix} f_{\mathbf{i}\downarrow} \\ -f_{\mathbf{i}\uparrow}^\dagger \end{pmatrix}$$

  - Compact Effective Lagrangian

$$L_1 = \frac{\tilde{J}}{2}\sum_{\langle \mathbf{ij}\rangle} \mathrm{Tr}\left[U_{\mathbf{ij}}^\dagger U_{\mathbf{ij}}\right] + \frac{\tilde{J}}{2}\sum_{\langle \mathbf{ij}\rangle,\sigma}\left(\Phi_{\mathbf{i}\sigma}^\dagger U_{\mathbf{ij}}\Phi_{\mathbf{j}\sigma} + \text{c.c.}\right) + \sum_{\mathbf{i}\sigma} f_{\mathbf{i}\sigma}^\dagger\left(\partial_\tau - i\lambda_{\mathbf{i}}\right) f_{\mathbf{i}\sigma}$$

$$+ \sum_{\mathbf{i}} b_{\mathbf{i}}^*\left(\partial_\tau - i\lambda_{\mathbf{i}} + \mu_B\right) b_{\mathbf{i}} - \sum_{\mathbf{ij}} t_{\mathbf{ij}} b_{\mathbf{i}} b_{\mathbf{j}}^* f_{\mathbf{i}\sigma}^\dagger f_{\mathbf{j}\sigma}$$

$$U_{\mathbf{ij}} = \begin{pmatrix} -\chi_{\mathbf{ij}}^* & \Delta_{\mathbf{ij}} \\ \Delta_{\mathbf{ij}}^* & \chi_{\mathbf{ij}} \end{pmatrix} \qquad \chi_{\mathbf{ij}} = \sum_\sigma \langle f_{\mathbf{i}\sigma}^\dagger f_{\mathbf{j}\sigma}\rangle \qquad \Delta_{\mathbf{ij}} = \langle f_{\mathbf{i}\uparrow} f_{\mathbf{j}\downarrow} - f_{\mathbf{i}\downarrow} f_{\mathbf{i}\uparrow}\rangle$$

  - Lagrangian invariant under $\Phi_{\mathbf{i}\sigma} \to W_{\mathbf{i}}\Phi_{\mathbf{i}\sigma} U_{\mathbf{ij}} \to W_{\mathbf{i}} U_{\mathbf{ij}} W_{\mathbf{j}}^\dagger$
  - Connecting mean field ansatz

$$U_{\mathbf{ij}}^{\pi\text{-flux}} = -\chi\left(\tau^3 - i(-1)^{i_x + j_y}\right) \qquad U_{\mathbf{i},i+\mu}^d = -\chi\left(\tau^3 + \eta_\mu \tau^1\right)$$

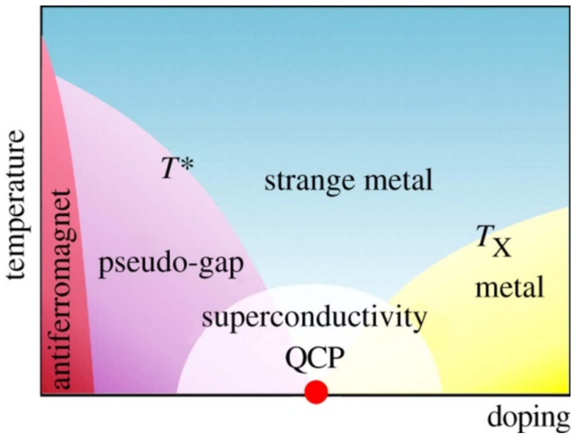
$$U_{\mathbf{ij}}^{SF} = W_{\mathbf{i}}^\dagger U_{\mathbf{ij}}^d W_{\mathbf{j}} \qquad W_{\mathbf{j}} = \exp\left[i(-1)^{j_x + j_y}\frac{\pi}{4}\tau^1\right] \qquad \Phi_{\mathbf{i}}' = W_{\mathbf{i}}\Phi_{\mathbf{i}}$$

  - Ground state of ➔ antiferromagnetic long range ordering (AFLRO)
  - Hence we can *naively* decouple the exchange interaction

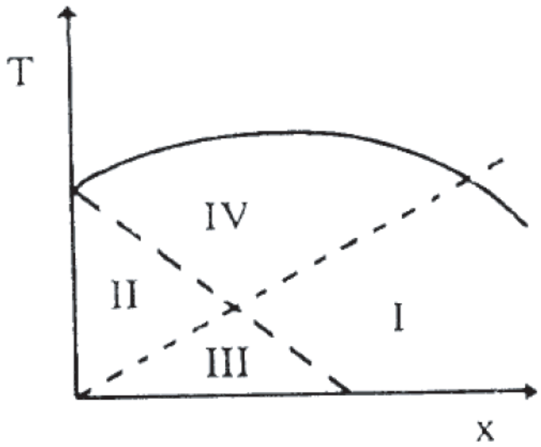
$$\mathbf{S_i}\cdot\mathbf{S_j} = \frac{1}{4} f_{\mathbf{i}\alpha}^\dagger \sigma_{\alpha\beta}^\mu f_{\mathbf{i}\beta} f_{\mathbf{j}\gamma}^\dagger \sigma_{\gamma\delta}^\mu f_{\mathbf{j}\delta}$$

VIII. Slave boson formulation of t-J model and mean field theory

- **The doped case**
  - Undoped ($x = 0$) only has spin dynamics
  - Bosons are crucial for charge dynamics
  - **No** Bose-Einstein condensation (BEC) in 2D!
  - *Weak* interlayer hole-hopping ➔ $T_{BE} \neq 0$
  - Slave boson model ➔ 5 phases classified by χ, Δ, and $b = \langle b_\mathbf{i} \rangle$



Label	State	χ	Δ	b
I	Fermi liquid	$\neq 0$	$= 0$	$\neq 0$
II	Spin gap	$\neq 0$	$\neq 0$	$= 0$
III	*d*-wave superconducting	$\neq 0$	$\neq 0$	$\neq 0$
IV	uRVB	$\neq 0$	$= 0$	$= 0$



Cuprates overview

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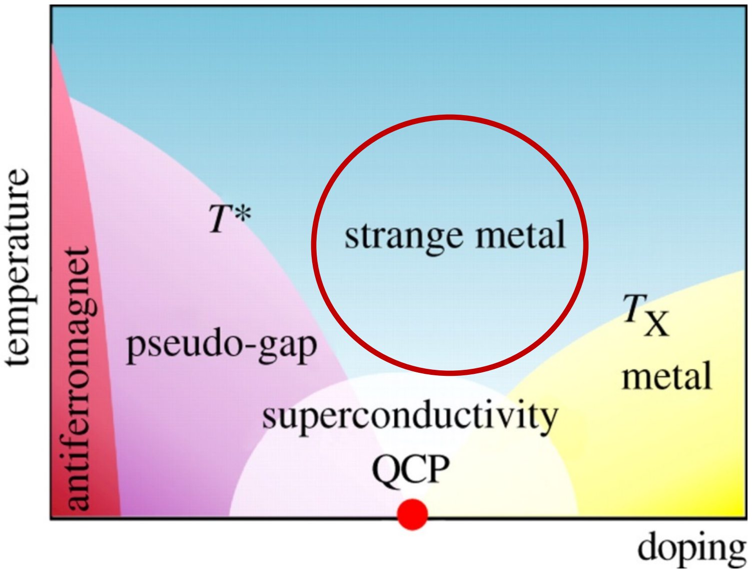
  - Pseudogap
  - Stripes
  - Nodal quasiparticles
- **Introduction to Resonating Valence Bond (RVB)**
- **Phase fluctuations vs. competing order**
- **Numerical techniques**
- **Single hole problem**
- **Slave particles and gauge fields**
  - Mean field theory
  - U(1) gauge theory
  - Confinement physics

IX. U(1) gauge theory of the RVB state

- **Motivation**
  - Mean field theory enforces no-double-occupany **on average**
  - Treat fluctuations about mean field on a Gaussian level
  - Redundancy of U(1) phase in defining fermion and boson
- **U(1) gauge theory**
  - Ioffe-Larkin composition rule
  - Describes high temperature limit of the optimally doped cuprate
- **Limitations**
  - Fails in the underdoped region
  - Gaussian theory also misses the confinement physics



IX. U(1) gauge theory of the RVB state

A. Effective gauge action and non-Fermi-liquid behavior

- **Slave-boson formalism**

  - "Fractionalizing" the electron

$$c^\dagger_{\mathbf{i}\sigma} = f^\dagger_{\mathbf{i}\sigma} b_{\mathbf{i}}$$

  - **Local** gauge degree of freedom

$$f_{\mathbf{i}\sigma} \to e^{\mathbf{i}\varphi_{\mathbf{i}}} f_{\mathbf{i}\sigma} \qquad b_{\mathbf{i}} \to e^{i\varphi_{\mathbf{i}}} b_{\mathbf{i}\sigma}$$

  - Fermion/boson strongly coupled to the gauge field ➔ conservation of the gauge charge

$$Q_{\mathbf{i}} = \sum_\sigma f^\dagger_{\mathbf{i}\sigma} f_{\mathbf{i}\sigma} + b^\dagger_{\mathbf{i}} b_{\mathbf{i}}$$

  - Green's functions transform as

$$G_F(\mathbf{i},\mathbf{j};\tau) \to e^{i(\varphi_{\mathbf{i}}-\varphi_{\mathbf{j}})} G_F(\mathbf{i},\mathbf{j};\tau) \qquad G_F(\mathbf{i},\mathbf{j};\tau) = -\langle T_\tau f_{\mathbf{i}\sigma}(\tau) f^\dagger_{\mathbf{j}\sigma}\rangle$$

$$G_B(\mathbf{i},\mathbf{j};\tau) \to e^{i(\varphi_{\mathbf{i}}-\varphi_{\mathbf{j}})} G_B(\mathbf{i},\mathbf{j};\tau) \qquad G_B(\mathbf{i},\mathbf{j};\tau) = -\langle T_\tau b_{\mathbf{i}}(\tau) b^\dagger_{\mathbf{j}}\rangle$$

  - Definition of gauge fields

$$a_{\mathbf{ij}} \to a_{\mathbf{ij}} + \varphi_{\mathbf{i}} - \varphi_{\mathbf{j}}$$

$$a_0(\mathbf{i}) \to a_0(\mathbf{i}) + \frac{\partial \varphi_{\mathbf{i}}(\tau)}{\partial \tau}$$

IX. U(1) gauge theory of the RVB state

A. Effective gauge action and non-Fermi-liquid behavior

- **Gaussian approximation**

  - Relevant Lagrangian

$$L_1 = \sum_{i,\sigma} f^*_{\mathbf{i}\sigma}\left(\frac{\partial}{\partial\tau} - \mu_F + i a_0(\mathbf{r_i})\right) f_{\mathbf{i}\sigma} + \sum_i b^*_{\mathbf{i}}\left(\frac{\partial}{\partial\tau} - \mu_B + i a_0(\mathbf{r_i})\right) b_{\mathbf{i}}$$
$$-\tilde{J}\chi \sum_{\langle \mathbf{ij}\rangle\sigma} \left(e^{i a_{\mathbf{ij}}} f^*_{\mathbf{i}\sigma} f_{\mathbf{j}\sigma} + \text{h.c.}\right) - t\eta \sum_{\langle \mathbf{ij}\rangle} \left(e^{i a_{\mathbf{ij}}} b^*_{\mathbf{i}} b_{\mathbf{j}} + \text{h.c.}\right)$$

  - $\mathbf{a}_{\mathbf{ij}} \rightarrow \mathbf{a}_{\mathbf{ij}} + 2\pi$ ➔ Lattice gauge theory coupled fermions/bosons

  - Gauge field has no dynamics ➔ coupling constant of the gauge field is infinity

  - Integrate out the matter fields: $e^{-S_{\text{eff}}(a)} = \int Df^* Df Db^* Db\, e^{-\int_0^\beta L_1}$

  - Gaussian approximation or RPA (continuum limit) $a_{\mathbf{ij}} = (\mathbf{r_i} - \mathbf{r_j}) \cdot \mathbf{a}\left(\frac{\mathbf{r_i} + \mathbf{r_j}}{2}\right)$

$$m_F \propto \frac{1}{J} \qquad m_B \propto \frac{1}{t}$$

$$L = \int d^2\mathbf{r} \left[\sum_\sigma f^*_\sigma(\mathbf{r})\left(\frac{\partial}{\partial\tau} - \mu_F + i a_0(\mathbf{r})\right) f_\sigma(\mathbf{r}) + b^*(\mathbf{r})\left(\frac{\partial}{\partial\tau} - \mu_B + i a_0(\mathbf{r})\right) b(\mathbf{r}) \right.$$
$$\left. - \frac{1}{2m_F} \sum_{\sigma, j=x,y} f^*_\sigma(\mathbf{r})\left(\frac{\partial}{\partial x_j} + i a_j\right)^2 f_\sigma(\mathbf{r}) - \frac{1}{2m_B} \sum_{j=x,y} b^*(\mathbf{r})\left(\frac{\partial}{\partial x_j} + i a_j\right)^2 b(\mathbf{r})\right]$$

IX. U(1) gauge theory of the RVB state

A. Effective gauge action and non-Fermi-liquid behavior

- **Effective gauge field action**

  - Gaussian Lagrangian

$$L = \int d^2\mathbf{r} \left[\sum_\sigma f_\sigma^*(\mathbf{r}) \left(\frac{\partial}{\partial \tau} - \mu_F + i a_0(\mathbf{r}) \right) f_\sigma(\mathbf{r}) + b^*(\mathbf{r}) \left(\frac{\partial}{\partial \tau} - \mu_B + i a_0(\mathbf{r}) \right) b(\mathbf{r}) \right.$$
$$\left. - \frac{1}{2m_F} \sum_{\sigma, j=x,y} f_\sigma^*(\mathbf{r}) \left(\frac{\partial}{\partial x_j} + i a_j \right)^2 f_\sigma(\mathbf{r}) - \frac{1}{2m_B} \sum_{j=x,y} b^*(\mathbf{r}) \left(\frac{\partial}{\partial x_j} + i a_j \right)^2 b(\mathbf{r}) \right]$$

  - Coupling between the matter fields and gauge field

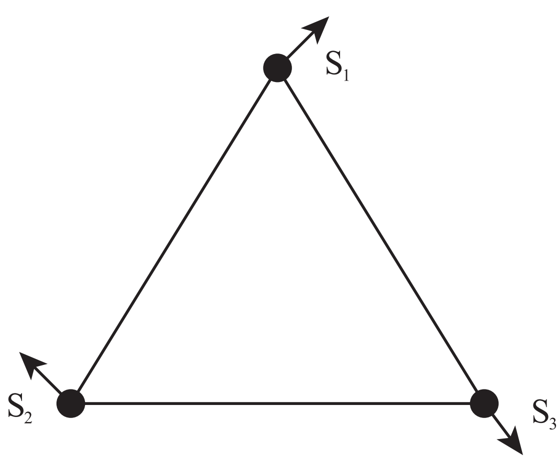
$$L_{\text{int}} = \int d^2\mathbf{r}\, (j_\mu^F + j_\mu^B) a_\mu$$

  - Constraints after integrating over temporal and spatial components of a_μ

$$f_{\mathbf{i}\uparrow}^\dagger f_{\mathbf{i}\uparrow} + f_{\mathbf{i}\downarrow}^\dagger f_{\mathbf{i}\downarrow} + b_{\mathbf{i}}^\dagger b_{\mathbf{i}} = 1$$
$$\mathbf{j}_F + \mathbf{j}_B = 0$$

  - Physical meaning of the gauge field? Consider electron moving in a loop

$$P_{123} = \langle \chi_{12}\chi_{23}\chi_{31} \rangle = \langle f_{1\alpha}^\dagger f_{2\alpha} f_{2\beta}^\dagger f_{3\beta} f_{3\gamma}^\dagger f_{1\gamma} \rangle$$
$$\frac{1}{4i}(P_{123} - P_{132}) = \mathbf{S}_1 \cdot (\mathbf{S}_2 \times \mathbf{S}_3)$$



IX. U(1) gauge theory of the RVB state

B. Ioffe-Larkin composition rule

- **Physical quantities in terms of fermions/bosons**
  - The fermion/boson current $\mathbf{j}_F = \sigma_F \mathbf{e}_F, \quad \mathbf{j}_B = \sigma_F \mathbf{e}_B$
  - External **E** field ➔ gauge field **a** induces "internal" electric field **e**

$$\mathbf{e}_F = \mathbf{E} + \mathbf{e} \qquad \mathbf{e}_B = \mathbf{e}$$

  - Recall constraint $\mathbf{j}_F + \mathbf{j}_B = 0$

$$\mathbf{e} = -\frac{\sigma_F}{\sigma_F + \sigma_B}\mathbf{E} \qquad \mathbf{j} = \mathbf{j}_F = -\mathbf{j}_B = \frac{\sigma_F \sigma_B}{\sigma_F + \sigma_B}\mathbf{E} \qquad \sigma = \frac{\sigma_B \sigma_F}{\sigma_B + \sigma_F}$$

  - "Scattering" from the gauge field

$$\frac{1}{\tau_{\mathrm{tr}}^B} \propto k_B T \quad \Rightarrow \sigma_B \propto \frac{n_B \tau_{\mathrm{tr}}^B}{m_B} \propto \frac{xt}{T} \qquad \sigma_F \propto J \Rightarrow \sigma_F \gg \sigma_B \Rightarrow \sigma \approx \sigma_B$$

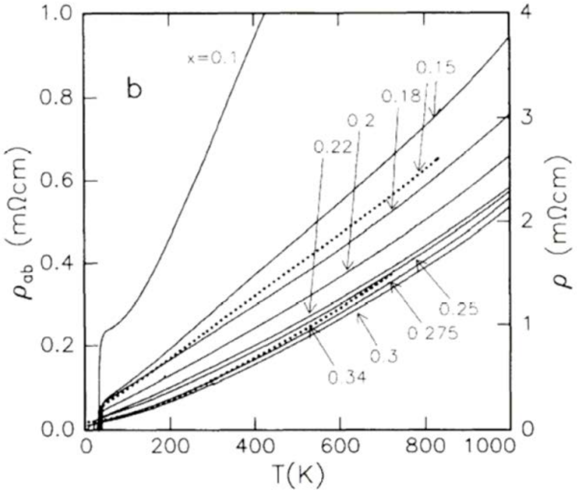
  - Temperature-dependent superfluid density

$$\rho = \frac{\rho_F \rho_B}{\rho_F + \rho_B} = \rho_B \left(1 + \frac{\rho_B}{\rho_F}\right)^{-1} \approx \rho_B \left(1 - \frac{\rho_B}{\rho_F}\right)$$

$$\rho_s(T) = \rho_s^B(0) - \frac{(\rho_s^B(0))^2}{\rho_s^F(0)} aT$$

$$R_H = \frac{R_H^F \chi_B + R_H^F \chi_F}{\chi_B + \chi_F}$$

$$\kappa = \kappa_B + \kappa_F$$



IX. U(1) gauge theory of the RVB state

B. Ioffe-Larkin composition rule

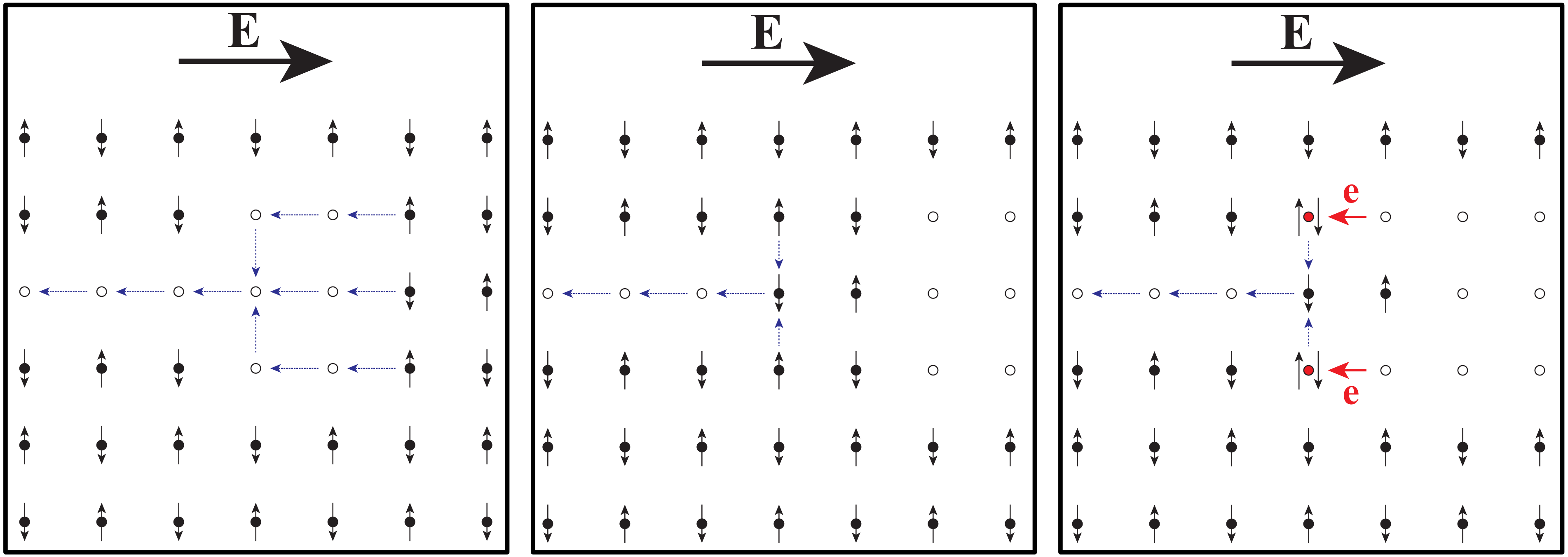
- **Physical effects of gauge field**
  - Physical conductivity

$$\sigma^{-1} = \sigma_F^{-1} + \sigma_B^{-1}$$

  - External **E** field ➔ gauge field **a** induces “internal” electric field **e**

$$\mathbf{e}_F = \mathbf{E} + \mathbf{e}$$

$$\mathbf{e}_B = \mathbf{e}$$



IX. U(1) gauge theory of the RVB state

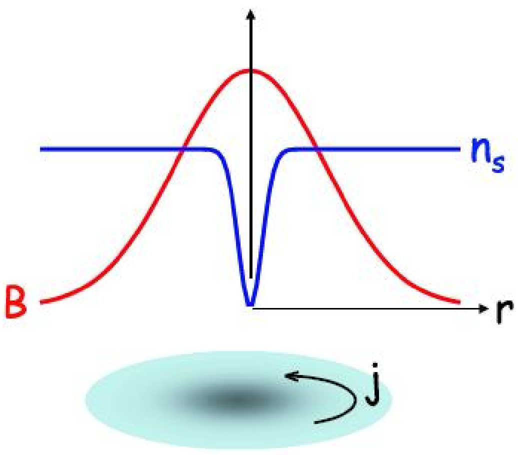
C. Ginzburg-Landau theory and vortex structure

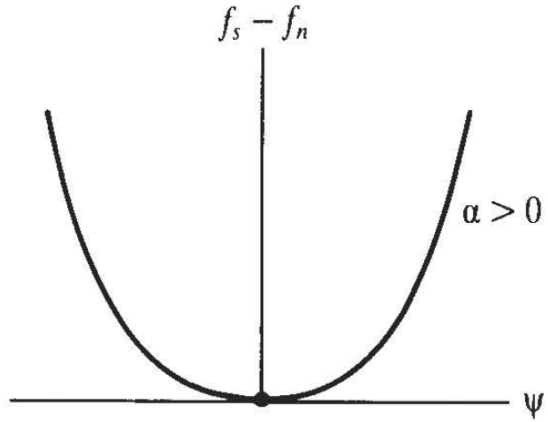
- **The Berezinskii-Kosterlitz-Thouless (BKT) Transition**
  - Free energy of a single CuO_2 layer $F = F_F[\psi, \mathbf{a}, \mathbf{A}] + F_B[\phi, \mathbf{a}] + F_{\text{gauge}}[\mathbf{a}]$

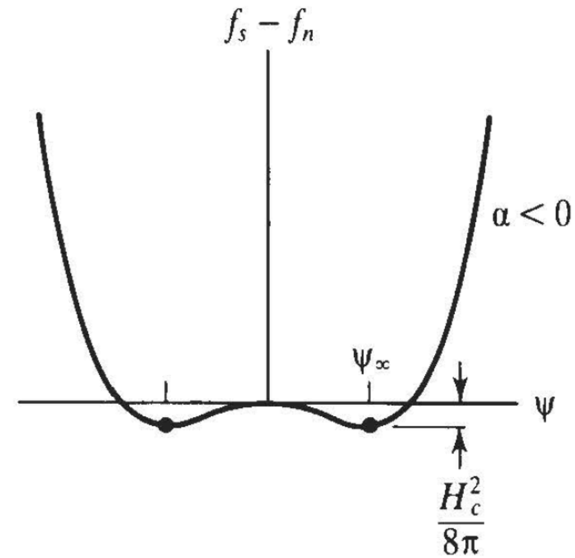
$$F_F[\psi, \mathbf{a}, \mathbf{A}] = \frac{H_{cF}^2}{8\pi} \int d^2\mathbf{r} \left[2\xi_F^2 \left| \left(\nabla - 2i\mathbf{a} - i\frac{2e}{c}\mathbf{A} \right) \psi \right|^2 + 2\text{sign}\left(T - T_D^{(0)} \right) |\psi|^2 + |\psi|^4 \right]$$

$$F_B[\psi, \mathbf{a}] = \frac{H_{cB}^2}{8\pi} \int d^2\mathbf{r} \left[2\xi_B^2 |(\nabla - i\mathbf{a})\phi|^2 + 2\text{sign}\left(T - T_{BE}^{(0)} \right) |\phi|^2 + |\phi|^4 \right]$$

$$F_{\text{gauge}}[\mathbf{a}] = \int d^2\mathbf{r} \left[\chi_F \left[\nabla \times \left(\mathbf{a} + \left(\frac{e}{c} \right) \mathbf{A} \right) \right]^2 + \chi_B \left(\nabla \times \mathbf{a} \right)^2 \right]$$



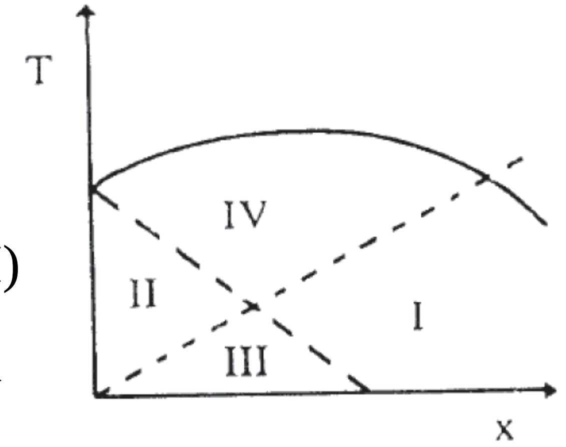




IX. U(1) gauge theory of the RVB state

C. Ginzburg-Landau theory and vortex structure

- **Vortex structure**
  - Type A: Vortex core state is the Fermi liquid (I)
  - Type B: Vortex core state is the spin gap state (II)
  - Energy contribution from region *far* away from the core ($> \xi_B, \xi_F$) for type A

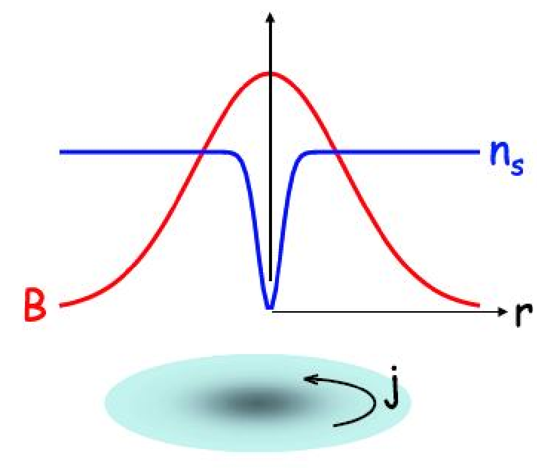


$$E_0 = \left[\frac{\phi_0}{4\pi\lambda}\right]^2 \ln\left[\frac{\lambda}{\max(\xi_F, \xi_B)}\right]$$

  - Condensation energy for types A and B:

$$E_c^{(A)} \approx H_{cF}^2 \xi_F^2 \approx \frac{\Delta^2}{J}\left(\frac{J}{\Delta}\right)^2 \approx J$$

$$E_c^{(B)} \approx H_{cB}^2 \xi_B^2 \approx t x^2 \left(\frac{1}{x^{1/2}}\right)^2 \approx t x$$



  - Total vortex energies

$$E^{(A)} \approx E_0 + E_c^{(A)} \qquad E^{(B)} = 4E_0 + E_c^{(B)} \qquad E_0 \propto \frac{1}{\lambda^2} \approx \frac{1}{\lambda_B^2} \approx x$$

Cuprates overview

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  - Mean field theory
  - U(1) gauge theory
  - Confinement physics

IX. U(1) gauge theory of the RVB state

D. Confinement-deconfinement problem

- **Pure lattice gauge theory**

  - Compact lattice gauge theory without matter field

$$S_{\text{gauge}} = -\frac{1}{g}\sum_{\text{plaquette}}(1-\cos(f_{\mu\nu}))$$

$$f_{\mu\nu} = a_{\mathbf{i},\mathbf{i}+\boldsymbol{\mu}} + a_{\mathbf{i}+\boldsymbol{\mu},\mathbf{i}+\boldsymbol{\mu}+\boldsymbol{\nu}} - a_{\mathbf{i}+\boldsymbol{\nu},\mathbf{i}+\boldsymbol{\mu}+\boldsymbol{\nu}} - a_{\mathbf{i},\mathbf{i}+\boldsymbol{\nu}}$$

  - Wilson loop as order parameter

$$W(C) = \left\langle \exp\left[iq\oint_C dx_\mu a_\mu(x)\right]\right\rangle$$

  - In terms of gauge potential

$$W(C) = \exp\left[-V(R)T\right]$$

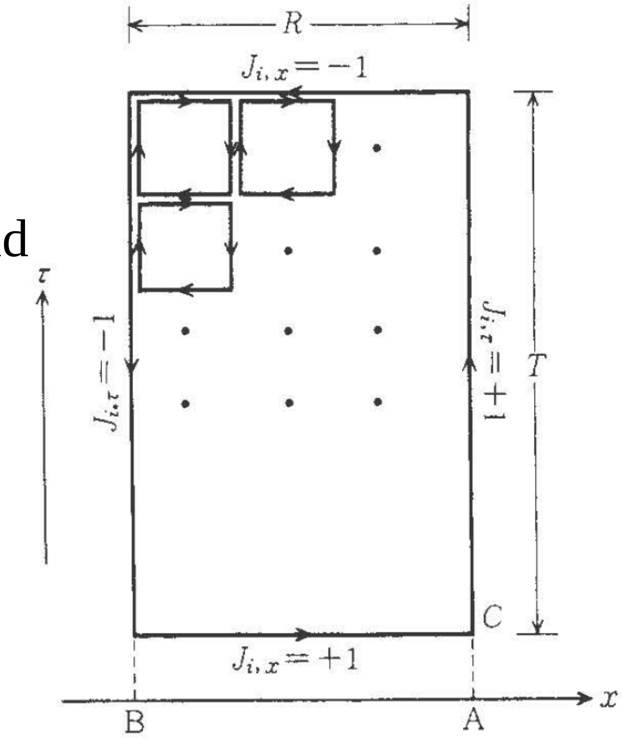
  - Area (confined) vs. perimeter (deconfined) law

$$W_{\text{confined}}(C) \sim e^{-\alpha RT} \qquad W_{\text{deconfined}}(C) \sim e^{-\beta(R+T)}$$

  - The "instanton" is the source of the gauge flux with the field distribution

$$\mathbf{b}(\mathbf{x}) = \frac{\mathbf{x}}{2|\mathbf{x}|^3} \qquad \mathbf{x} = (\mathbf{r},\tau) \qquad \mathbf{b}(\mathbf{x}) = (e_y(\mathbf{x}), -e_x(\mathbf{x}), b(\mathbf{x}))$$

  - Flux slightly above (future) or below (past) of the instanton differs by 2π

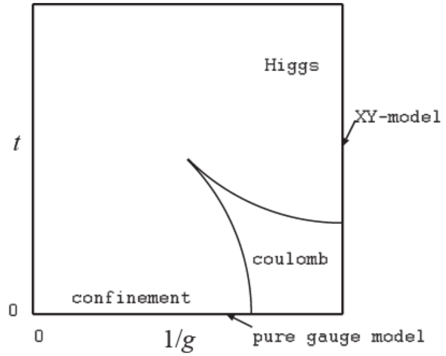


IX. U(1) gauge theory of the RVB state

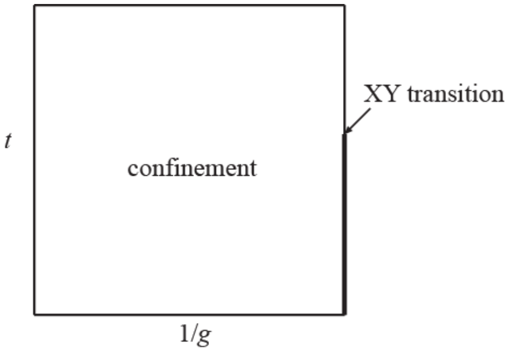
D. Confinement-deconfinement problem

- **Coupling of gauge theory to paired matter fields**
  - Is deconfined ground state possible in U(1) gauge theory?
  - Consider following bosonic field coupled to compact U(1) field (coupling constant g)

$$S_B = t \sum_{\mathbf{i}} \cos\left(\Delta_\mu \theta(\mathbf{r_i}) - q a_\mu(\mathbf{r_i})\right)$$



  - For $g \ll 1$, S_B reduces to an XY model weakly coupled to a U(1) gauge field (q = 1)
  - In (2+1)D t-g plane is covered by Higgs-confinement phase (q = 1)
  - If bosonic field is pairing field ➔ q = 2
  - Pairing implicitly has Z_2 gauge symmetry
  - **Quantum** Z_2 (Ising) gauge theory in 2D has a confinement-deconfinement transition



IX. U(1) gauge theory of the RVB state

D. Confinement-deconfinement problem

- **Coupling of gauge theory to gapless matter fields**
  - Is deconfined ground state possible in without pairing?
  - Yes, dissipation due to gapless excitations lead to deconfinement
  - This (gapless) U(1) spin liquid arises naturally from SU(2) formulation

- **Controversies on U(1) gauge theory confinement**
  - Nayak ➔ slave particles are always confined in U(1) gauge theories due to infinite coupling
  - Partially integrating out the matter fields makes coupling finite (but strong)
  - Several counter examples found to Nayak’s claim

IX. U(1) gauge theory of the RVB state

E. Limitations of the U(1) gauge theory

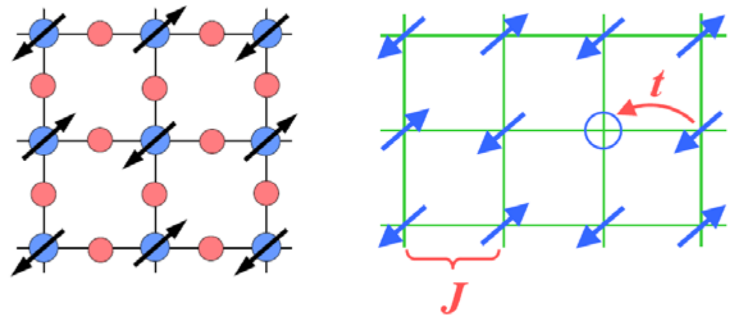
- **Discrepancies in temperature-dependent superfluid density**
  - In the Gaussian approximation, current carried by quasiparticles in the superconducting state is xv_F
  - Confinement leads to BCS-like quasiparticles carrying the full current
- **Cannot explain spin correlations at (π, π)**
  - Gauge field is gapped in the fermion paired state
  - Gauge fluctuations cannot account for enhanced spin correlations seen in neutron scattering at (π, π)
- **Energetically stable "*hc/e*" vortex not observed**
  - STM failed to see the *hc/e* vortex
  - U(1) theory misses the low lying fluctuations related to SU(2) particle-hole symmetry at half-filling

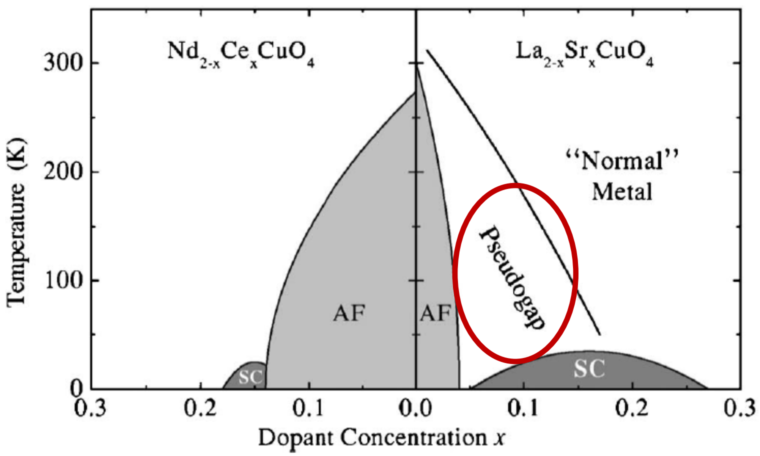
Summary of the cuprates (part 1)

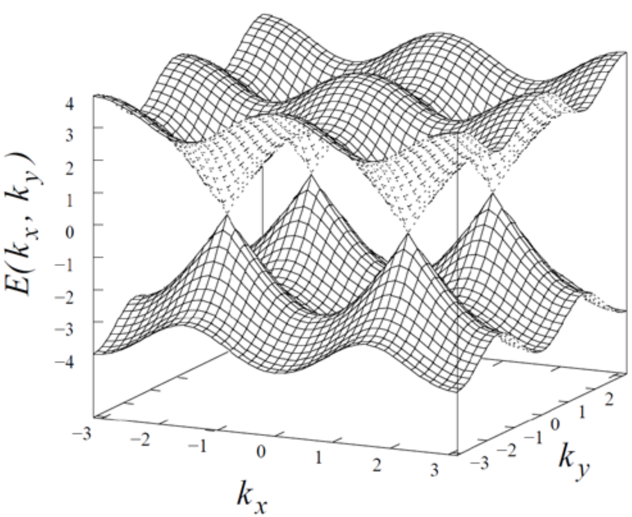
- **Electronic structure**
  - Relevant physics confined to 2D
  - The "t-J" model

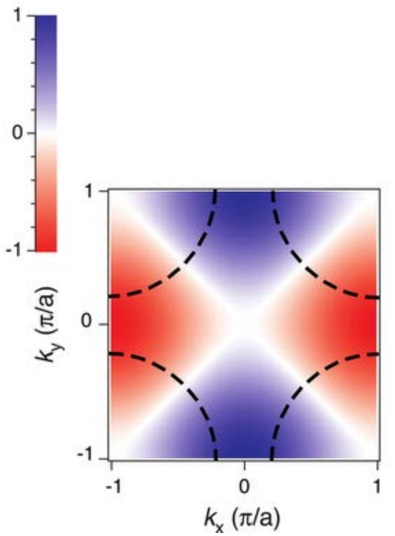
$$H = P\left[-\sum_{\langle \mathbf{ij}\rangle,\sigma} t_{\mathbf{ij}} c^{\dagger}_{\mathbf{i}\sigma} c_{\mathbf{i}\sigma} + J\sum_{\langle \mathbf{ij}\rangle}\left(\mathbf{S_i}\cdot\mathbf{S_j} - \frac{1}{2}n_{\mathbf{i}}n_{\mathbf{j}}\right)\right]P$$

  - Universal phase diagram
- **Phenomenology of the cuprates**
  - Experimental signatures of the pseudogap phase
  - Nodal quasiparticles
- **Slave bosons**
  - Slave fermions and bosons
  - U(1) gauge theory



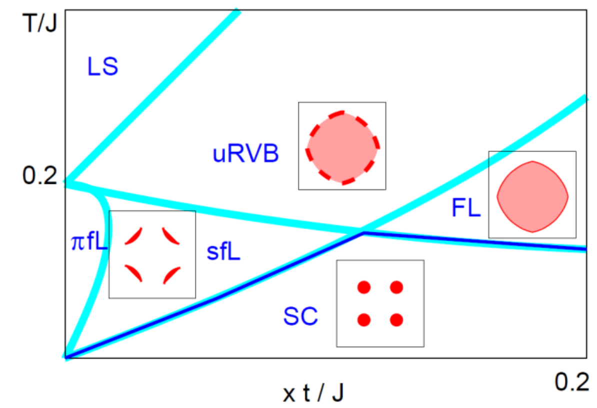


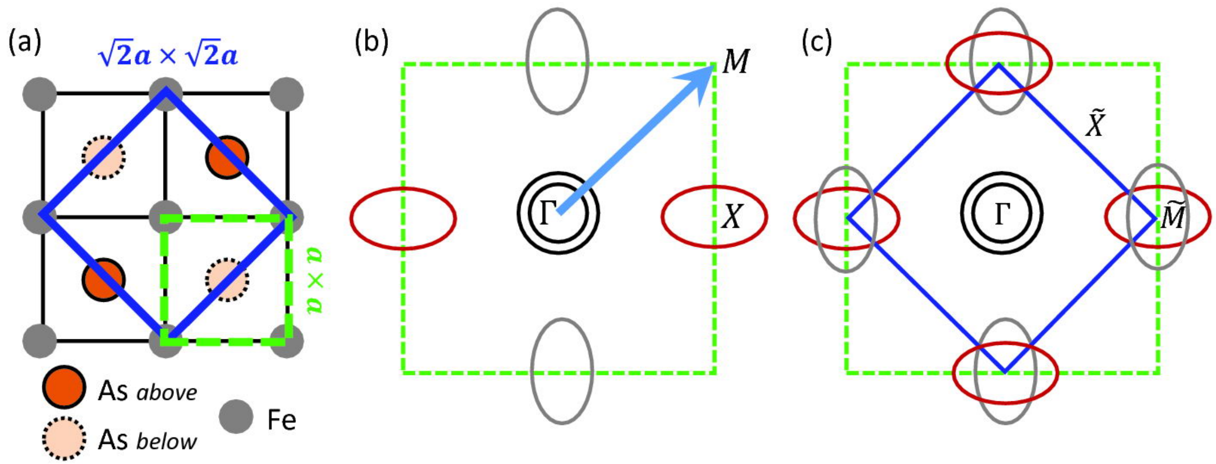


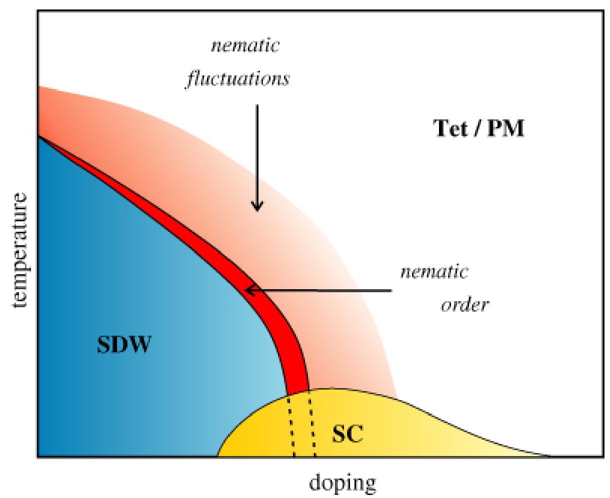


Next time: cuprates (cont'd) and pnictides

- **SU(2) slave boson theory at finite doping**
  - SU(2) theory gives richer phase diagram than U(1) theory
  - SU(2) theory captures confinement physics missed by U(1) theory
- **Iron-based (pnictide) superconductors**
  - Discovered in **2008** by Kamihara
  - Physics confined to 2D like cuprates
  - Pseudogap replaced by "nematic phase"







Thanks for listening!

Questions?