Fractional Quantum Hall Effect

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Recap: Chern-Simons Theory

The Lagrangian for the Chern-Simons theory can be written as

$$\mathcal{L}_{\mathrm{CS}} = \frac{m}{4\pi}\varepsilon^{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda + j^\mu a_\mu \tag{1}$$

where a^μ is the "internal" gauge field and j_μ is the current due to the quasiparticles. Nick showed that this theory has

- Topological order (ground state degeneracy, fractionalization)
- Quantized "Hall conductivity"

Outline

- Connecting this theory to the "external" gauge field $\equiv$ the EM field (i.e. connect to the physical QHE)
- Hierarchical construction and K-matrix theory
- Edge states

1 Connection to the physical quantum Hall effect (QHE)

- We want an effective theory which will work at low energies (complete theory too complicated)
- We guess the theory and then see if it is right by comparing it to other calculations
- The effective Lagrangian, with the Maxwell terms, as well as a guess $\mathcal{L}_{\mathrm{top}}[J]$ (which will later turn out to be the Chern-Simons term) can be written as

$$\mathcal{L} = F_{\mu\nu}F^{\mu\nu} + \frac{e}{2\pi}J^\mu A_\mu + \mathcal{L}_{\mathrm{top}}[J] \tag{2}$$

- Due to conservation of charge we know that $\partial_\mu J^\mu = 0$; hence we can write it in terms of an "internal" gauge field a_λ as

$$J^\mu = \frac{e}{2\pi}\varepsilon^{\mu\nu\lambda}\partial_\nu a_\lambda \tag{3}$$

Now, assuming (or guessing) the $\mathcal{L}_{\text{top}}[J]$ to be the Chern-Simons term, the full Lagrangian reads

$$\mathcal{L} = F_{\mu\nu}F^{\mu\nu} + \left(\frac{e}{2\pi}\right)^2 \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda + \frac{m}{4\pi}\varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda \tag{4}$$

Minimizing the Lagrangian with respect to a_μ we get

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial a_\mu} &= \left(\frac{e}{2\pi}\right)^2 \varepsilon^{\mu\nu\lambda} \partial_\nu A_\lambda + \frac{m}{4\pi}\varepsilon^{\mu\nu\lambda} \partial_\nu a_\lambda \\ &= 0\end{aligned} \tag{5}$$

Using the definition $B^\mu = \varepsilon^{\mu\nu\lambda}\partial_\nu A_\lambda$ as well as (3) and adding in the appropriate dimensional constants we recover the Hall conductivity

$$J = \left(\frac{1}{m}\right)\frac{e^2}{h}B \tag{6}$$

Consider the properties of the $a\partial a$ term: "Hopf term"

$$2\left(\frac{2\pi}{e}\right)^2 \pi m J_1^\mu \left(\frac{1}{\partial_\lambda \varepsilon^{\mu\nu\lambda}}\right) J_2^\nu = 2\left(\frac{2\pi}{e}\right)^2 \pi m \int d^3\mathbf{x}\; J_1^\mu f_\mu \tag{7}$$

where we have defined

$$f_\mu \equiv \left(\frac{1}{\partial_\lambda \varepsilon^{\mu\nu\lambda}}\right) J_2^\nu \tag{8}$$

If we let

$$\begin{aligned} J_1 &= \delta(\mathbf{x}_1 - \mathbf{x}) \\ J_2 &= \delta(0) \end{aligned}$$

such that path of J_1 encircles J_2. Then (7) becomes

$$\begin{aligned} 2\left(\frac{2\pi}{e}\right)^2 \pi m \int d^3\mathbf{x}\; J_1^\mu f_\mu &= 2\left(\frac{2\pi}{e}\right)^2 \pi m \oint_C f_\mu \\ &= 2\left(\frac{2\pi}{e}\right)^2 \pi m \int_S \partial f \\ &= 2\left(\frac{2\pi}{e}\right)^2 \pi m \int_S \delta(0) \\ &= 2\left(\frac{2\pi}{e}\right)^2 \pi m \end{aligned} \tag{9}$$

This is just saying that J particles have fermionic statistics (which makes sense since they are fermions). Also, in the second line we have used the fact that

$$\partial^\mu f_\mu = J_2 \tag{10}$$

In summary, the action in terms of free electrons:

- has reproduced the fractional quantum Hall effect
- has **not** reproduced the fractional charge, fractional statistics, and the more complicated fractions

2 Quasiparticles

In order to recover the fractional charge and fractional statistics we can add the term $lj^\mu a_\mu$, where j^μ is the quasiparticle current which carries the charge of a field. Now the Lagrangian becomes

$$\mathcal{L} = F_{\mu\nu}F^{\mu\nu} + \left(\frac{e}{2\pi}\right)^2 \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda + \frac{m}{4\pi}\varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + lj^\mu a_\mu \tag{11}$$

Integrating out all the a_μ degrees of freedom, while interchanging $\partial_\nu \leftrightarrow k_\mu$, we end up with an effective Lagrangian given by

$$\mathcal{L} = \frac{\pi l^2}{m} j^\mu \left(\frac{1}{\partial_\lambda \varepsilon^{\mu\nu\lambda}}\right) j^\nu + \frac{el}{m} j^\mu A_\mu + \frac{e^2}{m}\varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + \text{Maxwell term} \tag{12}$$

From each of the terms on the RHS we get:

- $\Theta = \dfrac{2\pi l^2}{m}$
- $q = \dfrac{el}{m}$
- Chern-Simons term in the external fields
- The Maxwell terms are irrelevant $\Rightarrow$ coupling has mass dimension -1; and we know that terms with mass dimension -1 are irrelevant and low energies
- $a\partial a$ is the most relevant term allowed by symmetry and gauge invariance

3 More complicated fractions (Hierarchical construction)

Let us define a new $\tilde{a}_\mu$ such that

$$j^\mu = \varepsilon^{\mu\nu\lambda}\partial_\nu \tilde{a}_\lambda \tag{13}$$

Then we can write down the overall Lagrangian is

$$\mathcal{L} = \frac{e}{2\pi}\varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda - \frac{m_1}{4\pi}\varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \frac{1}{2\pi}\varepsilon^{\mu\nu\lambda} a_\mu \partial_\nu \tilde{a}_\lambda - \frac{m_2}{4\pi}\varepsilon^{\mu\nu\lambda} \tilde{a}_\mu \partial_\nu \tilde{a}_\lambda \tag{14}$$

- j^μ is a boson, so m_2 is even
- We can minimize the above Lagrangian with respect to a_μ and $\tilde{a}_\mu$ to get the Hall conductivity
- We can integrate out a_μ and $\tilde{a}_\mu$ to get the fractional charge and fractional statistics

However, we do not need to do these manipulations every time; this has been elegantly worked out in the K-matrix theory. We can write down (14) more compactly as

$$\mathcal{L} = -\frac{1}{4\pi}\varepsilon_{\mu\nu\lambda} K_{IJ} a_I^\mu \partial^\nu a_J^\lambda + \frac{e}{2\pi} q_I A_\mu \partial a_I^\mu \tag{15}$$

where

$$
\begin{aligned}
K_{IJ} &= \begin{bmatrix} m_1 & -1 \\ -1 & m_2 \end{bmatrix} \\
q_I &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\end{aligned}
\tag{16}
$$

Excitations can carry the charge of a_μ and $\tilde{a}_\mu$: $j' l_I a_I$. The standard FQHE quantities can be determined as

$$
\sigma = q^T K^{-1} q \tag{17}
$$

$$
Q = e q^T K^{-1} \tag{18}
$$

$$
\Theta = \pi l^T K^{-1} l \tag{19}
$$

- We can thus reproduce all experimentally observable fractions
- We can also deal with multilayer states this way

4 Edge States

- The Chern-Simons term is not gauge invariant; we'll claim that they are physical
- Pick them to give the right results
- Try $a_0 = 0$, $a_i = \partial_i \phi$

$$
\begin{aligned}
S &= -\frac{m}{4\pi} \int dx \; \partial_t \left(\partial_x \phi\right) \left(\partial_y \phi\right) \\
&= -\frac{m}{4\pi} \int dx \; \partial_t \phi \partial_x \phi
\end{aligned}
\tag{20}
$$

  Note that the $\partial_y \phi$ term won't matter across the boundary. However, if $H = 0$, velocity is zero; this can't be right

- Try $a_0 = -v a_x$

$$
S = -\frac{m}{4\pi} \int dx \; \partial_t \phi \partial_x \phi + v \left(\partial_x \phi\right)^2 \tag{21}
$$

  Thus our Hamiltonian is

$$
H = v \left(\partial_x \phi\right)^2 \tag{22}
$$

  after canonically quantizing the Hamiltonian we get

$$
H = 2\pi m v \sum_k \rho_k^\dagger \rho_k \tag{23}
$$

  where the operators ρ_k satisfy the "Kac-Moody algebra"

$$
\left[\rho_k, \rho_{k'}\right] = \frac{k \delta_{k+k'}}{2\pi m} \tag{24}
$$

  only works when $vm < 0$

- More generally, we can write the edge action as

$$
S = \frac{1}{4\pi} \int dx \; \left[K_{IJ} \partial_t \phi_I \partial_x \phi_J - V_{IJ} \partial_x \phi_I \partial_x \phi_J\right] \tag{25}
$$