Criticality and entanglement in random quantum systems

Journal Club

Criticality and entanglement in random quantum systems

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Outline

Field guide to infinite-randomness fixed points

- Strong Disorder Renormalization Group
- Heisenberg chains: spin ½, spin 1, FM couplings
- Transverse Field Ising Chain
- Anyonic chains: Majorana Fermions, Fibonacci Anyons

Next week: entanglement entropy in infinite –randomness fixed points

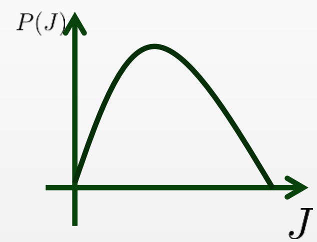
- …….

Real Space Renormalization Group

The simplest application is the spin ½ Heisenberg chain

Spin ½ Heisenberg chain

$$\mathcal{H} = \sum_i J_i \mathbf{S}_i \cdot \mathbf{S}_{i+1} \qquad S_i = 1/2$$



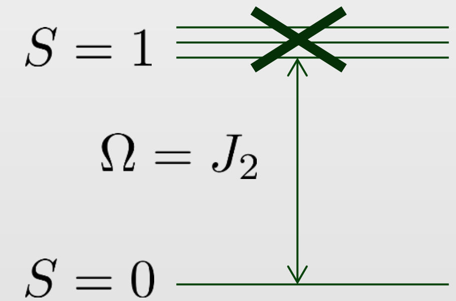
To derive the RG rules, it is sufficient to consider 4 spins

$$H = H_0 + H_1$$

Breaking the Hamiltonian:

Largest coupling $\left\{ H_0 = J_2 \mathbf{S}_2 \cdot \mathbf{S}_3 \right.$

Perturbation $\left\{ H_1 = J_1 \mathbf{S}_1 \cdot \mathbf{S}_2 + J_3 \mathbf{S}_3 \cdot \mathbf{S}_4 \right.$

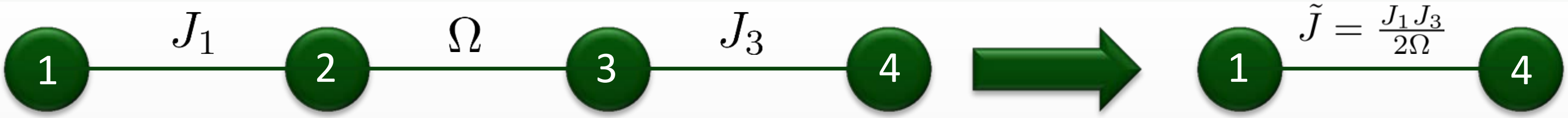


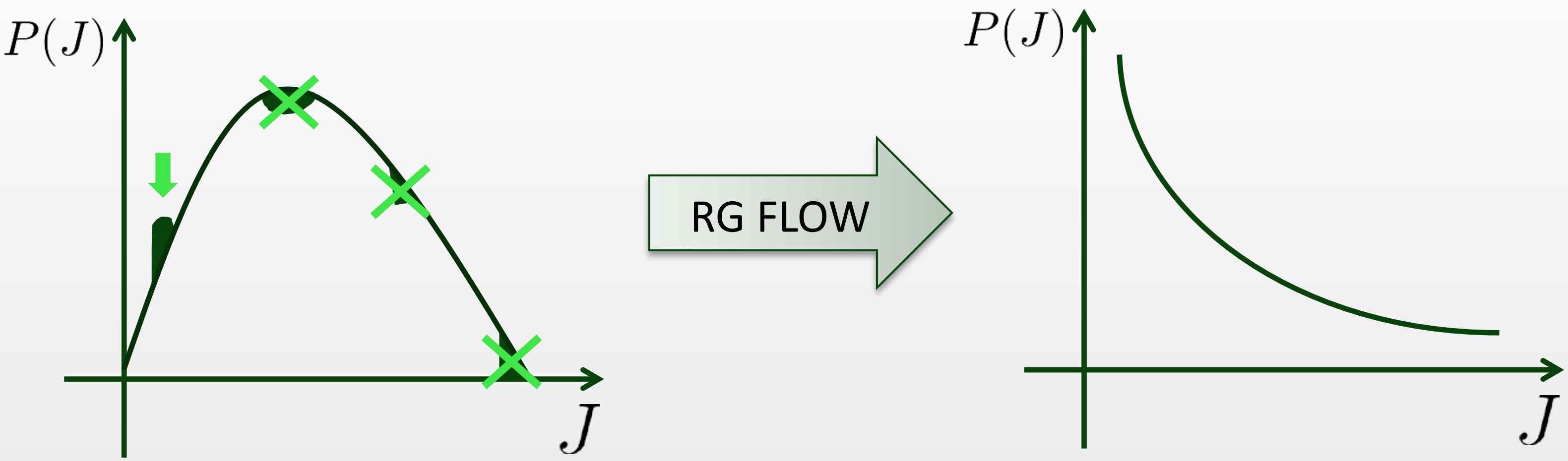
Second order perturbation theory:

Effective Hamiltonian $\tilde{H} = \tilde{J} \mathbf{S}_1 \cdot \mathbf{S}_4 \qquad \tilde{J} = \frac{J_1 J_3}{2\Omega}$

- S.-k. Ma, C. Dasgupta, and C.-k. Hu, Phys. Rev. Lett. 43, 1434 (1979).

Spin ½ Heisenberg chain





It is convenient to introduce log variables $\alpha = \frac{1}{\Gamma} = \frac{1}{\ln\left(\frac{\Omega_0}{\Omega}\right)}$

$$\xi = \ln\left(\frac{\Omega}{J}\right) \qquad \Gamma = \ln\left(\frac{\Omega_0}{\Omega}\right)$$

$\xi = \xi_1 + \xi_3$ ~~$-\,2$~~

- C. Dasgupta and S.-k. Ma, Phys. Rev. B 22, 1305 (1980).
- D. S. Fisher, Phys. Rev. B 50, 3799 (1994).

Assuming strong disorder, the master equation for $\rho\left(\xi, \Gamma\right)$ is

$$\frac{\partial \rho}{\partial \Gamma}\left(\xi, \Gamma\right) = \frac{\partial \rho}{\partial \xi}\left(\xi, \Gamma\right) + \rho_0 \int_0^{\xi} d\xi_1 \rho\left(\xi_1, \Gamma\right) \rho\left(\xi - \xi_1, \Gamma\right)$$

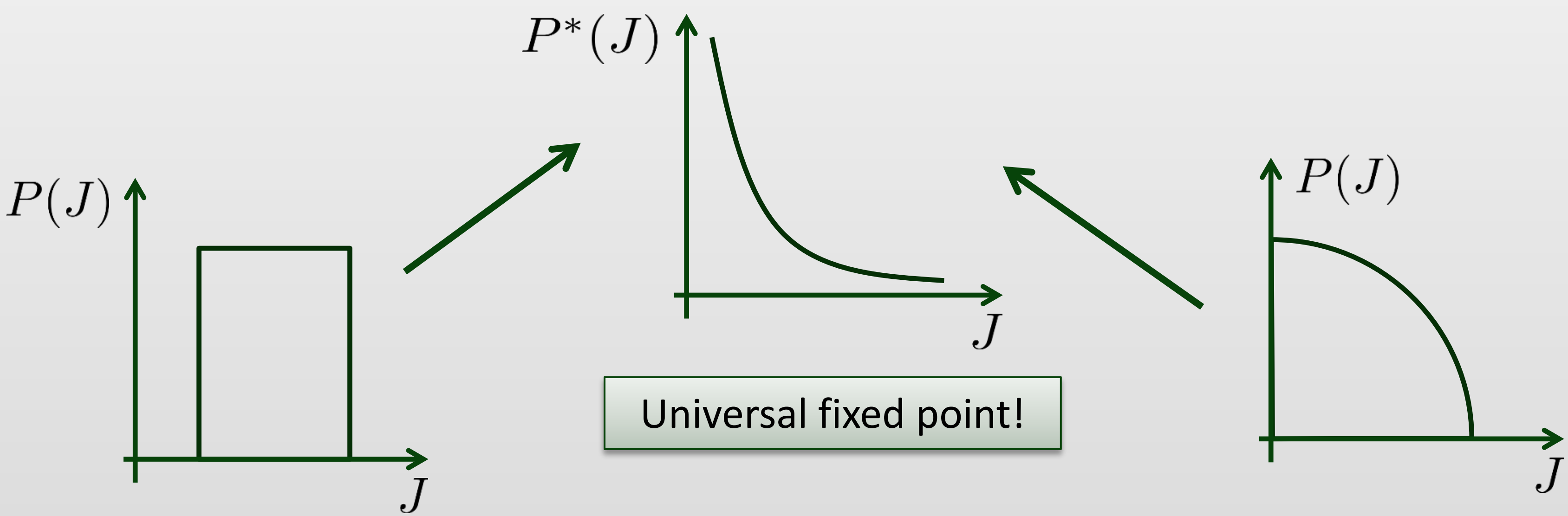
Spin ½ Heisenberg chain

$$\frac{\partial \rho}{\partial \Gamma}(\xi,\Gamma) = \frac{\partial \rho}{\partial \xi}(\xi,\Gamma) + \rho_0 \int_0^{\xi} d\xi_1 \rho(\xi_1,\Gamma)\,\rho(\xi-\xi_1,\Gamma)$$

This equation can be solved analytically!

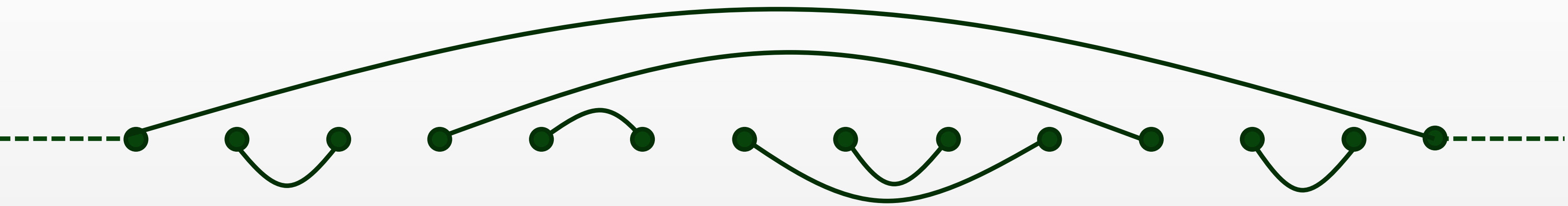
$$\rho^*(\xi,\Gamma) = \frac{\theta(\xi)}{\Gamma} e^{-\frac{\xi}{\Gamma}}$$

In terms of the original variables $P^*(J,\Omega) = \frac{\alpha}{\Omega}\left(\frac{\Omega}{J}\right)^{1-\alpha} \theta(\Omega - J)$



Spin ½ Random Singlet Phase

Ground State: collection of strong correlated singlets, some of them are far apart



Infinite disorder fixed point: $\lim_{\Omega\to 0} \frac{\sqrt{\mathrm{Var}\,J}}{\langle J\rangle} \propto \left(\frac{1}{\alpha}\right)^{1/2} \to \infty$

- The RG is **asymptotically exact**
- Disorder is **relevant**
- The fixed point is **universal** and **stable**

Spin ½ Heisenberg chain

$$P^*(J,\Omega) = \frac{\alpha}{\Omega}\left(\frac{\Omega}{J}\right)^{1-\alpha}\theta(\Omega - J)$$

Once we have the fixed-point distribution, we can compute the energy-length scaling

Energy-length scaling

$$dn_\Gamma = -2n_\Gamma \rho(0,\Gamma)\, d\Gamma$$
$$\frac{dn_\Gamma}{n_\Gamma} = -2\frac{d\Gamma}{\Gamma}$$

$$\ln n_\Gamma = -2\ln\Gamma$$

$$n_\Gamma = \frac{1}{\Gamma^{\frac{1}{\psi}}}$$

$$\psi = \frac{1}{2}$$

Thermodynamics: Stop the flow when $T = \Omega$

Magnetic Susceptibility

$$\chi \sim \frac{n_{\Omega=T}}{T} \sim \frac{1}{T\left[\ln\left(\frac{1}{T}\right)\right]^{1/\psi}}$$

Entropy

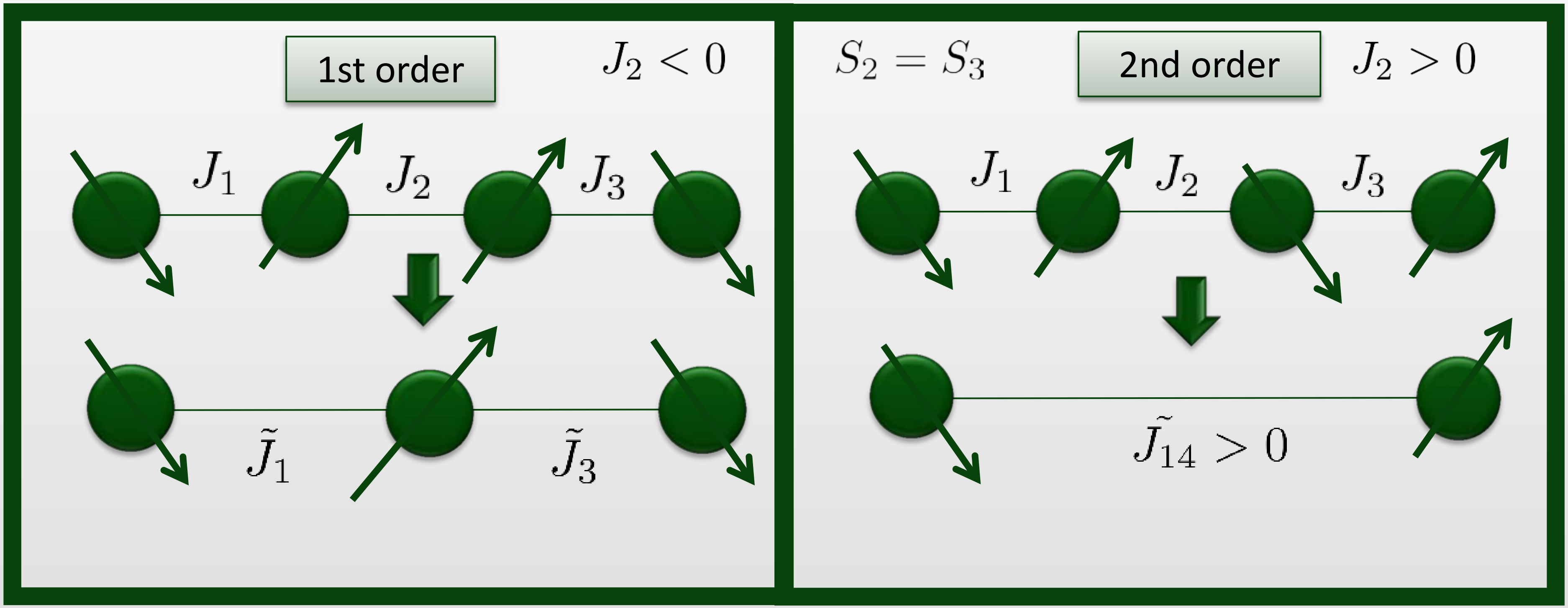
$$S \propto k_B n_{T=\Omega} \ln 2 \propto \frac{1}{\left[\ln\left(\frac{1}{T}\right)\right]^{1/\psi}}$$

Specific Heat

$$c = T\frac{dS}{dT} \sim \frac{1}{\left[\ln\left(\frac{1}{T}\right)\right]^{1+1/\psi}}$$

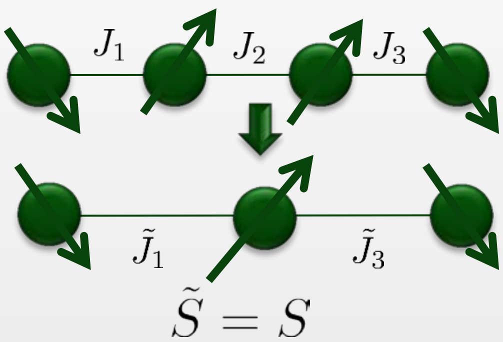
Ferromagnetic couplings

In the case where there is a finite fraction of FM couplings, the average spin size increases



Ferromagnetic couplings

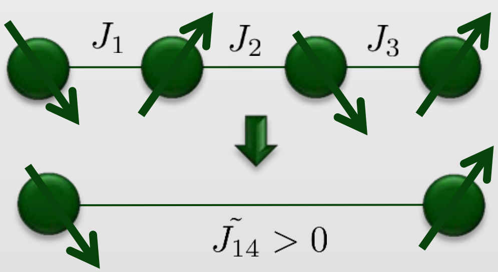
The RG rules are



1st order

$$\tilde{J}_1 = \frac{S(S+1)+S_2(S_2+1)-S_3(S_3+1)}{2S(S+1)} J_1$$

$$\tilde{J}_3 = \frac{S(S+1)+S_3(S_3+1)-S_2(S_2+1)}{2S(S+1)} J_3$$

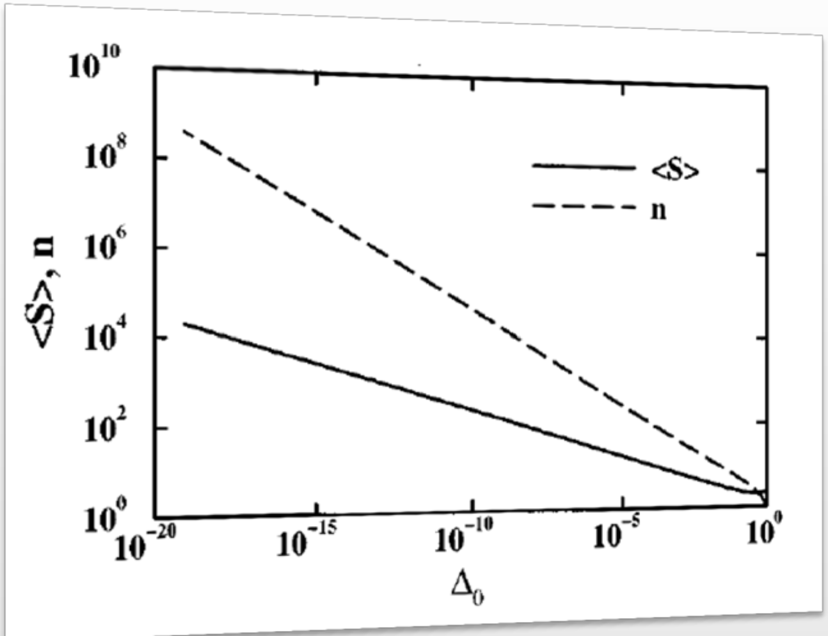


2nd order

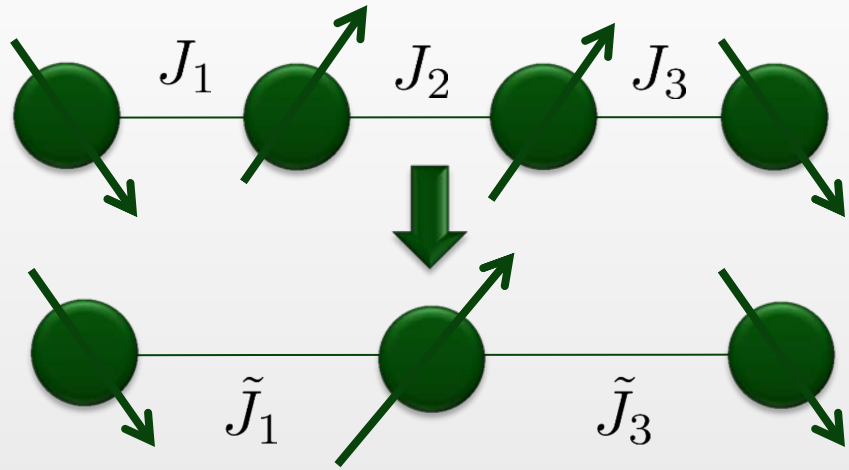
$$\tilde{J}_{14} = \frac{2 J_1 J_3 S_2(S_2+1)}{3 J_2}$$

- E. Westerberg, A. Furusaki, M. Sigrist, and P. A. Lee, Phys. Rev. B 55, 12578 (1997).

Ferromagnetic couplings



- E. Westerberg, A. Furusaki, M. Sigrist, and P. A. Lee, Phys. Rev. B 55, 12578 (1997).



$$S = \left| \sum_{i=1}^{l} \mp S_i \right|$$

Random Walk

$$S \sim L_\Omega^{1/2}$$

Both the AF and FM chains are unstable to couplings of different sign

Physical properties with FM and AF couplings

z is non-universal

Energy-length scale

$$\Omega \sim L^{-z}$$

Entropy

$$\sigma = k_B n_\Omega \ln\left(2\left\langle S_{eff}\right\rangle + 1\right) \propto T^{\frac{1}{z}} \left|\ln T\right|$$

Specific Heat

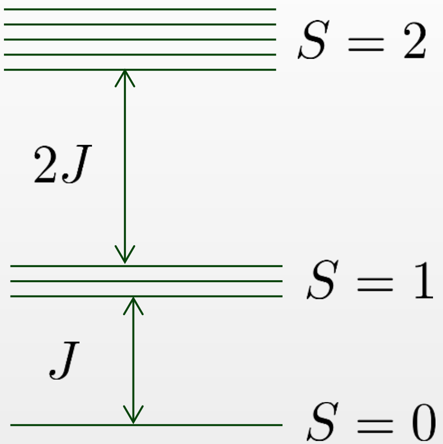
$$C = T\frac{d\sigma}{dT} \propto T^{\frac{1}{z}} \left|\ln T\right|$$

Magnetic susceptibility

$$\chi = \frac{\mu^2 n_\Omega \left\langle S_{eff}^2\right\rangle}{3k_B T} = \frac{c}{T}$$

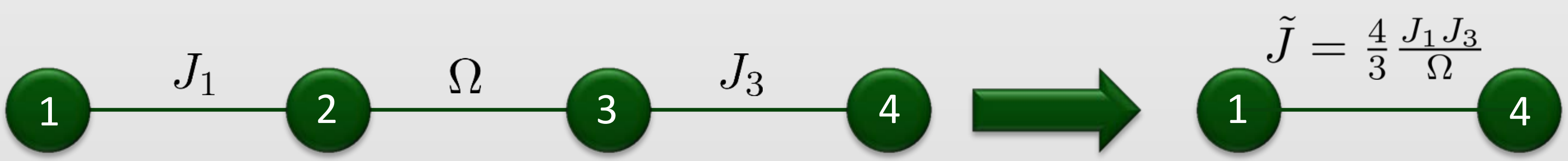
Spin-1 Heisenberg Chain

$$\mathcal{H} = \sum_i J_i \mathbf{S}_i \cdot \mathbf{S}_{i+1} \qquad S_i = 1$$



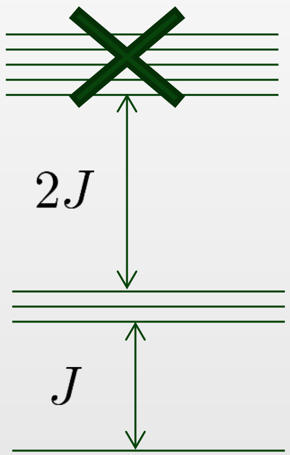
How to generalize to larger spins?

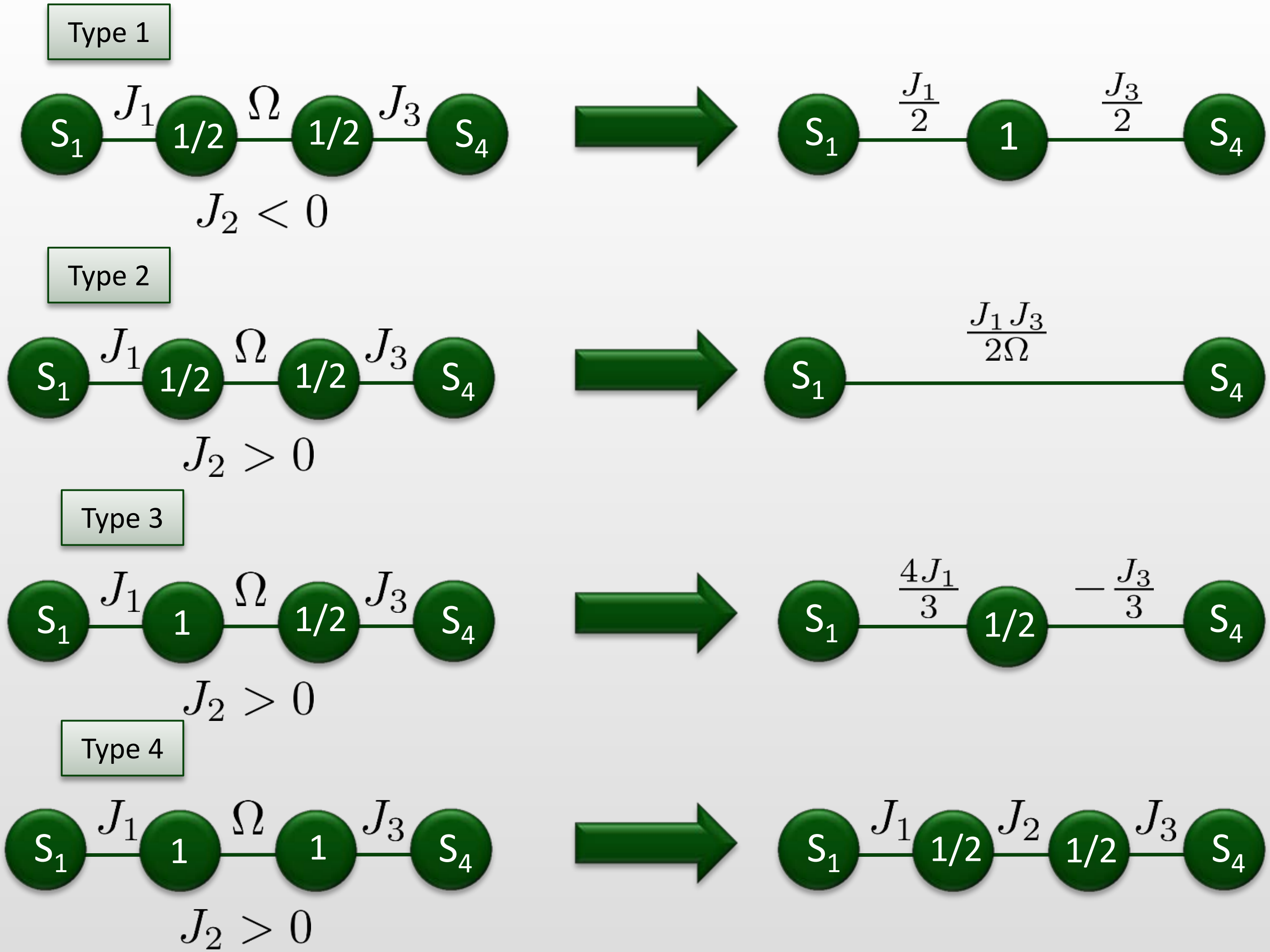
If we keep just the S=0 state, the RG rule is



For weak disorder, perturbation theory breaks down!

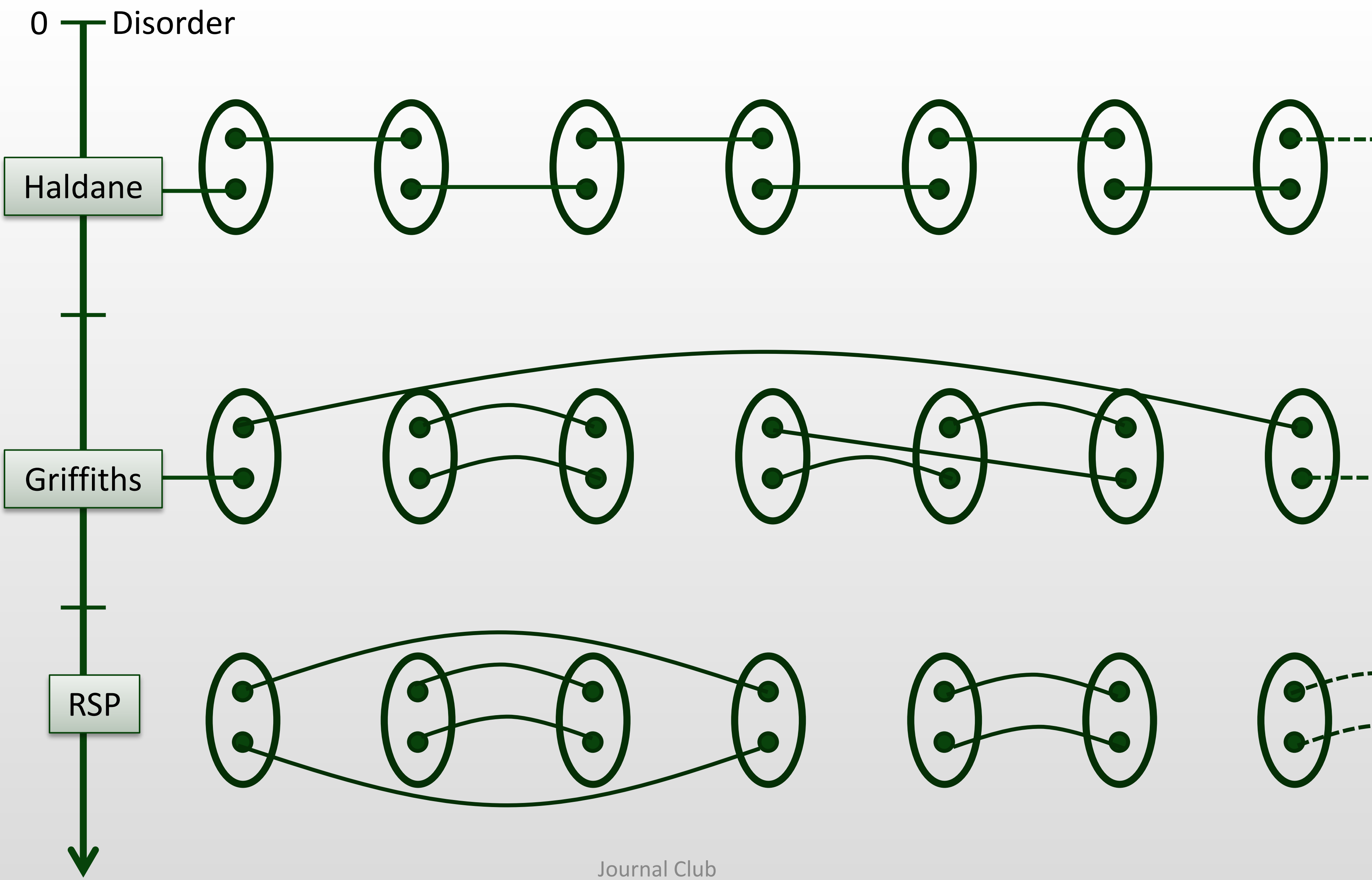
Spin 1 chain: weak disorder





- C. Monthus, O. Golinelli, and T. Jolicoeur, Phys. Rev. B **58**, 805 (1998), Phys. Rev. Lett. **79**, 3254 (1997)

Phases diagram for the spin-1 Heisenberg chain



Transverse Field Ising Chain (TFIC)

$$H = -\sum_i \left(J_i \sigma_i^z \sigma_{i+1}^z + h_i \sigma_i^x \right)$$

$h/J \gg 1$: Paramagnetic phase

$h/J \ll 1$: Ferromagnetic phase

Clean system: correlation length exponent $\nu = 1$

$$\xi \sim \frac{1}{|h - J|}$$

Defining $\Omega = \max\{J_i, h_i\}$ and the log variables $\beta = \ln\left(\frac{\Omega}{h}\right)$ and $\xi = \ln\left(\frac{\Omega}{J}\right)$

$$\frac{\partial P(\beta)}{\partial \Gamma} = \frac{\partial P(\beta)}{\partial \beta} + R(0) \int d\beta_1 \int d\beta_3 \delta(\beta - \beta_1 - \beta_3) P(\beta_1) P(\beta_2) + P(\beta)(P(0) - R(0))$$

$$\frac{\partial R(\xi)}{\partial \Gamma} = \frac{\partial P(\xi)}{\partial \xi} + P(0) \int d\xi_1 \int d\xi_3 \delta(\xi - \xi_1 - \xi_3) R(\xi_1) R(\xi_2) + R(\xi)(R(0) - P(0))$$

Transverse Field Ising Chain (TFIC)

$$\frac{\partial P(\beta)}{\partial \Gamma} = \frac{\partial P(\beta)}{\partial \beta} + R\left(0\right) \int d\beta_1 \int d\beta_3 \delta\left(\beta - \beta_1 - \beta_3\right) P\left(\beta_1\right) P\left(\beta_2\right) + P\left(\beta\right)\left(P\left(0\right) - R\left(0\right)\right)$$

$$\frac{\partial R(\xi)}{\partial \Gamma} = \frac{\partial P(\xi)}{\partial \xi} + P\left(0\right) \int d\xi_1 \int d\xi_3 \delta\left(\xi - \xi_1 - \xi_3\right) R\left(\xi_1\right) R\left(\xi_2\right) + R\left(\xi\right)\left(R\left(0\right) - P\left(0\right)\right)$$

Ansatz: $P\left(\beta\right) = be^{-b\beta}$ $R\left(\xi\right) = ge^{-g\xi}$

$$g = -\delta + \delta \cot \delta\Gamma$$
$$b = \delta + \delta \cot \delta\Gamma$$

$$\delta = \frac{\langle \ln J\rangle - \langle \ln h\rangle}{\sqrt{\mathrm{var}(\ln J) + \mathrm{var}(\ln h)}}$$

Critical point: $\delta = 0$ $b = g = \frac{1}{\Gamma}$

$$dn = -n\left(b + g\right) d\Gamma = -n d\Gamma 2\delta \coth\left(\delta\Gamma\right)$$

$$n = \frac{n_0 \delta^2}{\sinh^2 \delta\Gamma}$$ ⟹ $$L \sim \frac{1}{n} = E^{-2|\delta|} \implies z = \frac{1}{2|\delta|}$$

Non-Abelian anyons

- SU(2) fusion algebra $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$
- For FM coupling, arbitrary large spins are generated: the fusion algebra is not closed
- For non-Abelian systems, the rules are closed: truncated SU(2)

Properties

- Individual Hilbert spaces can have fractional dimension
- Prefactors are less than 1: can apply Ma-Dasgupta procedure safely!
- Infinite randomness behavior

Majorana fermions

- Fusion rule: $\sigma \otimes \sigma = 1 \oplus \psi$
- d=$\sqrt{2}$
- Hamiltonian: $H = \sum_j J_j i \sigma_j \sigma_{j+1}$
- Can be mapped to the TFIC by a Wigner-Jordan transformation

Non-Abelian anyons

Fibonacci anyons

- SU(2) algebra $\tau \otimes \tau = 1 \oplus \tau$ $\quad d = \frac{1}{2}\left(1+\sqrt{5}\right)$
- Fixed point distribution with ‘AF’ and ‘FM’ couplings:

$$N\left(\beta\right) = \frac{1}{\Gamma} e^{-2\beta/\Gamma}$$