Topology of Interacting Phases

Introduction

- We know of a lot of topological phases, but we want more!
- To get more, try including interactions
- Sometimes new topological phases appear
- Sometimes we find that phases that we though were distinct actually aren't (e.g. class BDI, $d = 1$, $\mathbb{Z} \to \mathbb{Z}_8$)

Short-ranged vs. long-ranged entanglement

The situation strongly depends on the type of entanglement

Short-ranged	Long ranged
Mutual information decays exponentially	Mutual information decays as a constant
Trivial if all symmetry broken	Toric code if all symmetry broken
No topological order	Topological order
'Symmetry Protected Topological Phase'	'Symmetry Enriched Topological Phase'

Entanglement details

Mutual conditional information
$S(A:B|C) = S(AC) + S(BC) - S(C) - S(ABC)$
Mutual information $S(A:B) = S(A) + S(B) - S(AB)$
Conditional Information $S(A|B) = S(AB) - S(B)$
$S(A) = \alpha L - \gamma$

SRE states

- Have interesting edge states
- Are otherwise trivial
- In one or two dimensions, either break symmetry or gapless (zero) mode
- In higher dimensions, can have other options (topological order or holographic state)
- At least in one dimension, can completely classify phases based on edge states (Wen)

Classifying all 1D phases

Can't use integral over Brillouin zone like we did for free fermions
Example: $SO(3)$ symmetry

- System must be invariant if symmetry applied to all of it
- But ends need not behave this way: can break symmetry e.g. spin $1/2$ on end
- Total spin is integer, so ends are either both $1/2$ or both 1
- Therefore two phases with different topology

Rigorous definition

- States defined by how the edges transform under symmetry
- Break the edge symmetry operator into two parts: L and R
- $g_i|\psi\rangle = L_{g_i} R_{g_i}|\psi\rangle$
- L_{g_i}, R_{g_i} only defined up to a phase: $L_{g_i} R_{g_i} = (L_{g_i} e^{i\alpha})(R_{g_i} e^{-i\alpha})$
- $L_{g_i} R_{g_i}$ is a representation of g, $L_{g_i} R_{g_i} L_{g_j} R_{g_j} = L_{g_i g_j} R_{g_i g_j}$
- But L_{g_i} by itself might not be: $L_{g_i} L_{g_j} = e^{i\rho(g_i, g_j)} L_{g_i, g_j}$
- The possible values for ρ classify the phases, but they are only defined up to the phases ϕ
- $\rho'(g_i, g_j) = \rho(g_i, g_j) + \alpha(g_i) + \alpha(g_j) - \alpha(g_i g_j)$
- $L_a L_b L_c = e^{i\rho(a,b)} L_{ab} L_c = e^{i(\rho(a,b) + \rho(ab,c))} L_{abc} \rightarrow$
  $\rho(g_i g_j, g_k) + \rho(g_i, g_j) = \rho(g_i, g_j g_k) + \rho(g_j, g_k)$

Simple example: $\mathbb{Z}_2$

Group elements e, x

$$\rho(e,e), \rho(e,x), \rho(x,e), \rho(x,x) \tag{1}$$
$$\rho(e,e) = \rho(e,g) = \rho(g,e) \tag{2}$$
$$\rho(e,e)' = \rho(e,e) + \alpha(e) \tag{3}$$
$$\rho(g,g)' = \rho(g,g) + 2\alpha(g) - \alpha(e) \tag{4}$$

Therefore any phases can be mapped into any other phases

Nontrivial example: $\mathbb{Z}_2 \times \mathbb{Z}_2$

- Group operations are e, x, y, z and identity $(xy = z)$
- Using results of previous slide, can set $\rho(e, A) = \rho(A, A) = 0$
- Using Eq.(22), $\rho(x, y) = \rho(y, x) = \rho(x, z)$...
- $2\rho(x, y) = -\rho(x, x) - \rho(y, y) - \rho(z, z) = 0$
- Therefore $\rho(x, y) = 0$ or π.
- Different gauges can make e.g. $\rho(x, x)$ non-zero, but will always have 2 choices for $\rho(x, y)$

Applications

- Spin 1 has SO(3) symmetry, but lets call it $Z_2 \times Z_2$
- We find that edges have $Z_2 \times Z_2 \times Z_2 \approx SU(2)$ when $\rho(x, y) = \pi$
- Therefore edges are spin-1/2
- Given a model, the α are set but you can compute all the ρ and see if they are in the same class

Tables

- Need a bigger table than for free fermions because momentum not conserved (e.g. four coupled chains)
- Can get 1D table using above method, related to second cohomology group
- Fermions with 'no symmetry' actually have fermion parity conservation, so there can be a topological phase (Majorana chain)
- $Z \to Z_8$: Fermion parity+time reversal, for free fermions can have any number of Majoranas
- With interactions, can turn 4 Majoranas into a fermion, and 2 fermions into a boson

2 Dimensions

- Topological order now also possible
- Cohomology classification works for most cases, but some (Kitaev) think it is incomplete
- Rest of section focuses on two examples of SRE phases of bosons
- Kitaev E_8 phase
- Bosonic Quantum Hall effect

K-matrix theory

- A way of organizing an abelian Chern-Simons theory
- Example: 2 layer ordinary Integer Quantum Hall effect
- Degeneracy on torus$=|\det K|$
- Statisitics of quasiparticles: $2\pi m$, where m is diagonal element
- Edge mode directions: signs of eigenvalues
- Can also get fractional charges, Hall conductivity
- K matrix description gives you an *effective theory*

Kitaev E_8 phase

- Need a phase with no topological order, only bosonic excitations and gapless edge states
- Lu & Vishwanath: Can always Higgs modes in different directions unless there is a symmetry
- Therefore we need a matrix with all diagonal elements even, $|detK| = 1$, all eigenvalues with same sign
- Smallest matrix that does this has dimension 8
- This is inconsistent with the cohomology classification!

Bosonic Quantum Hall Effect

- $U(1)$ symmetry forbids some Higgs terms, so we are allowed to have modes in different directions
- $K = \sigma_x$
- Senthil & Levin: A bosonic action with 'flux attachment' could realize such an effective theory
- My paper: Action in terms of bosons with local interactions which realizes this K matrix

Relation to topological order

- Duality between Ising model and Z_2 gauge theory
- Adding SRE to Ising model is equivalent to adding topological order (semions) to gauge theory
- Topologically ordered phases of semions can be classfied by their fusion rules
- Leads to classification of SRE phases which agrees with Wen

Future directions

- Classification of SRE phases in higher dimensions (i.e. a scheme which includes E_8 phase)
- More effective theories and microscopic models
- Understanding surface states in 3 dimensions (Xie Chen's talk: topologically ordered surface states)
- Classification of fermionic SRE phases
- Topological order