3. Second Chern number and its physical consequences

B. TRI topological insulators based on lattice Dirac models

- The continuum Dirac model in (4+1)-d dimensions is expressed as

$$H = \int d^4\mathbf{x}\left[\psi^\dagger(\mathbf{x})\Gamma^i\left(-i\partial_i\right)\psi(\mathbf{x}) + m\psi^\dagger(\mathbf{x})\Gamma^0\psi(\mathbf{x})\right]$$

- **Note**: the above model bears superficial resemblance to (3+1)-d *relativistic* Dirac model. Here we don't need a "5-vector" since we don't require Lorentz invariance. The gamma matrices satisfy the Clifford algebra $\{\Gamma^\mu, \Gamma^\nu\} = 2\delta_{\mu\nu}\mathbb{I}$
- The lattice (tight-binding) version of this model is written as

$$H = \sum_{\mathbf{n},\mathbf{i}}\left[\psi_\mathbf{n}^\dagger\left(\frac{c\Gamma^0 - i\Gamma^i}{2}\right)\psi_{\mathbf{n}+\mathbf{i}} + \text{h.c.}\right] + m\sum_\mathbf{n}\psi_\mathbf{n}^\dagger\Gamma^0\psi_\mathbf{n} \qquad i \equiv (i_0, \mathbf{i}) \qquad i = (1, \hat{\mathbf{x}})\,,\ (2, \hat{\mathbf{y}})\,,\ (3, \hat{\mathbf{z}})\,,\ (4, \hat{\mathbf{w}})$$

- To get the **k**-space we do the same old Wannier to Bloch transformation defined by $\psi_\mathbf{n} = \sum_\mathbf{k} e^{-i\mathbf{k}.\mathbf{n}}\psi_\mathbf{k}$

$$H = \sum_{\mathbf{n},i}\left[\left\{\sum_{\mathbf{k}'} e^{i\mathbf{k}'.\mathbf{n}}\psi_{\mathbf{k}'}^\dagger\right\}\left(\frac{c\Gamma^0 - i\Gamma^{i_0}}{2}\right)\left\{\sum_\mathbf{k} e^{-i\mathbf{k}.(\mathbf{n}+\mathbf{i})}\psi_\mathbf{k}\right\} + \text{h.c.}\right] + m\sum_\mathbf{n}\left\{\sum_{\mathbf{k}'} e^{i\mathbf{k}'.\mathbf{n}}\psi_{\mathbf{k}'}^\dagger\right\}\Gamma^0\left\{\sum_\mathbf{k} e^{-i\mathbf{k}.\mathbf{n}}\psi_\mathbf{k}\right\}$$

$$= \sum_{\mathbf{n},i}\left[\sum_{\mathbf{k}',\mathbf{k}} e^{i\mathbf{k}'.\mathbf{n}}e^{-i\mathbf{k}.(\mathbf{n}+\mathbf{i})}\psi_{\mathbf{k}'}^\dagger\left(\frac{c\Gamma^0 - i\Gamma^{i_0}}{2}\right)\psi_\mathbf{k} + \text{h.c.}\right] + m\sum_\mathbf{n}\sum_{\mathbf{k}',\mathbf{k}} e^{i\mathbf{k}'.\mathbf{n}}e^{-i\mathbf{k}.\mathbf{n}}\psi_{\mathbf{k}'}^\dagger\Gamma^0\psi_\mathbf{k}$$

$$= \sum_\mathbf{n}\left[\sum_{\mathbf{k}',\mathbf{k}} e^{-i(\mathbf{k}-\mathbf{k}').\mathbf{n}}\left\{\sum_\mathbf{i} e^{-i\mathbf{k}.\mathbf{i}}\right\}\psi_{\mathbf{k}'}^\dagger\left(\frac{c\Gamma^0}{2}\right)\psi_\mathbf{k} + \text{h.c.}\right] + \sum_\mathbf{n}\left[\sum_{\mathbf{k}',\mathbf{k}} e^{-i(\mathbf{k}-\mathbf{k}').\mathbf{n}}\psi_{\mathbf{k}'}^\dagger\left\{\sum_i e^{-i\mathbf{k}.\mathbf{i}}\left(\frac{-i\Gamma^{i_0}}{2}\right)\right\}\psi_\mathbf{k} + \text{h.c.}\right] +$$

$$m\sum_\mathbf{n}\sum_{\mathbf{k}',\mathbf{k}} e^{-i(\mathbf{k}-\mathbf{k}').\mathbf{n}}\psi_{\mathbf{k}'}^\dagger\Gamma^0\psi_\mathbf{k}$$

$$= \sum_{\mathbf{k}',\mathbf{k}}\left\{\sum_\mathbf{i} e^{-i\mathbf{k}.\mathbf{i}}\right\}\psi_{\mathbf{k}'}^\dagger\left(\frac{c\Gamma^0}{2}\right)\psi_\mathbf{k}\left\{\sum_\mathbf{n} e^{-i(\mathbf{k}-\mathbf{k}').\mathbf{n}}\right\} + \sum_{\mathbf{k}',\mathbf{k}}\psi_{\mathbf{k}'}^\dagger\left\{\sum_i e^{-i\mathbf{k}.\mathbf{i}}\left(\frac{-i\Gamma^{i_0}}{2}\right)\right\}\psi_\mathbf{k}\left\{\sum_\mathbf{n} e^{-i(\mathbf{k}-\mathbf{k}').\mathbf{n}}\right\} +$$

$$\text{h.c.} + m\sum_{\mathbf{k}',\mathbf{k}}\psi_{\mathbf{k}'}^\dagger\Gamma^0\psi_\mathbf{k}\left\{\sum_\mathbf{n} e^{-i(\mathbf{k}-\mathbf{k}').\mathbf{n}}\right\}$$

3. Second Chern number and its physical consequences

B. TRI topological insulators based on lattice Dirac models

- To get the **k**-space we do the same old Wannier to Bloch transformation defined by $\psi_{\mathbf{n}} = \sum_{\mathbf{k}} e^{-i\mathbf{k}.\mathbf{n}}\psi_{\mathbf{k}}$

$$H = \sum_{\mathbf{n},i}\left[\left\{\sum_{\mathbf{k}'} e^{i\mathbf{k}'.\mathbf{n}}\psi^{\dagger}_{\mathbf{k}'}\right\}\left(\frac{c\Gamma^0 - i\Gamma^{i_0}}{2}\right)\left\{\sum_{\mathbf{k}} e^{-i\mathbf{k}.(\mathbf{n}+\mathbf{i})}\psi_{\mathbf{k}}\right\} + \text{h.c.}\right] + m\sum_{\mathbf{n}}\left\{\sum_{\mathbf{k}'} e^{i\mathbf{k}'.\mathbf{n}}\psi^{\dagger}_{\mathbf{k}'}\right\}\Gamma^0\left\{\sum_{\mathbf{k}} e^{-i\mathbf{k}.\mathbf{n}}\psi_{\mathbf{k}}\right\}$$

$$= \sum_{\mathbf{n},i}\left[\sum_{\mathbf{k}',\mathbf{k}} e^{i\mathbf{k}'.\mathbf{n}} e^{-i\mathbf{k}.(\mathbf{n}+\mathbf{i})}\psi^{\dagger}_{\mathbf{k}'}\left(\frac{c\Gamma^0 - i\Gamma^{i_0}}{2}\right)\psi_{\mathbf{k}} + \text{h.c.}\right] + m\sum_{\mathbf{n}}\sum_{\mathbf{k}',\mathbf{k}} e^{i\mathbf{k}'.\mathbf{n}} e^{-i\mathbf{k}.\mathbf{n}}\psi^{\dagger}_{\mathbf{k}'}\Gamma^0\psi_{\mathbf{k}}$$

$$= \sum_{\mathbf{n}}\left[\sum_{\mathbf{k}',\mathbf{k}} e^{-i(\mathbf{k}-\mathbf{k}').\mathbf{n}}\left\{\sum_{\mathbf{i}} e^{-i\mathbf{k}.\mathbf{i}}\right\}\psi^{\dagger}_{\mathbf{k}'}\left(\frac{c\Gamma^0}{2}\right)\psi_{\mathbf{k}} + \text{h.c.}\right] + \sum_{\mathbf{n}}\left[\sum_{\mathbf{k}',\mathbf{k}} e^{-i(\mathbf{k}-\mathbf{k}').\mathbf{n}}\psi^{\dagger}_{\mathbf{k}'}\left\{\sum_{i} e^{-i\mathbf{k}.\mathbf{i}}\left(\frac{-i\Gamma^{i_0}}{2}\right)\right\}\psi_{\mathbf{k}} + \text{h.c.}\right] +$$
$$m\sum_{\mathbf{n}}\sum_{\mathbf{k}',\mathbf{k}} e^{-i(\mathbf{k}-\mathbf{k}').\mathbf{n}}\psi^{\dagger}_{\mathbf{k}'}\Gamma^0\psi_{\mathbf{k}}$$

$$= \sum_{\mathbf{k}',\mathbf{k}}\left\{\sum_{\mathbf{i}} e^{-i\mathbf{k}.\mathbf{i}}\right\}\psi^{\dagger}_{\mathbf{k}'}\left(\frac{c\Gamma^0}{2}\right)\psi_{\mathbf{k}}\left\{\sum_{\mathbf{n}} e^{-i(\mathbf{k}-\mathbf{k}').\mathbf{n}}\right\} + \sum_{\mathbf{k}',\mathbf{k}}\psi^{\dagger}_{\mathbf{k}'}\left\{\sum_{i} e^{-i\mathbf{k}.\mathbf{i}}\left(\frac{-i\Gamma^{i_0}}{2}\right)\right\}\psi_{\mathbf{k}}\left\{\sum_{\mathbf{n}} e^{-i(\mathbf{k}-\mathbf{k}').\mathbf{n}}\right\} +$$
$$\text{h.c.} + m\sum_{\mathbf{k}',\mathbf{k}}\psi^{\dagger}_{\mathbf{k}'}\Gamma^0\psi_{\mathbf{k}}\left\{\sum_{\mathbf{n}} e^{-i(\mathbf{k}-\mathbf{k}').\mathbf{n}}\right\}$$

$$\delta_{\mathbf{k},\mathbf{k}'} = \sum_{\mathbf{n}} e^{-i(\mathbf{k}-\mathbf{k}').\mathbf{n}}$$

$$H = \sum_{\mathbf{k}}\left\{\sum_{\mathbf{i}} e^{-i\mathbf{k}.\mathbf{i}}\right\}\psi^{\dagger}_{\mathbf{k}}\left(\frac{c\Gamma^0}{2}\right)\psi_{\mathbf{k}} + \sum_{\mathbf{k}}\psi^{\dagger}_{\mathbf{k}}\left\{\sum_{i} e^{-i\mathbf{k}.\mathbf{i}}\left(\frac{-i\Gamma^{i_0}}{2}\right)\right\}\psi_{\mathbf{k}} + \text{h.c.} + m\sum_{\mathbf{k}}\psi^{\dagger}_{\mathbf{k}}\Gamma^0\psi_{\mathbf{k}}$$

- The sum over i can be evaluated as $\sum_{\mathbf{i}} e^{-i\mathbf{k}.\mathbf{i}} + \text{h.c.} = 2\sum_{i_0}\cos(k_{i_0})$ $\quad \sum_{i} e^{-i\mathbf{k}.\mathbf{i}}\left(\frac{-i\Gamma^{i_0}}{2}\right) + \text{h.c.} = 2\sum_{i_0}\sin(k_{i_0})\Gamma^{i_0}$

- Diagonalized Hamiltonian in **k**-space

$$H = \sum_{\mathbf{k}}\psi^{\dagger}_{\mathbf{k}}\left[\sum_{i_0}\sin(k_{i_0})\Gamma^{i_0} + \left(m + c\sum_{i_0}\cos(k_{i_0})\right)\Gamma^0\right]\psi_{\mathbf{k}}$$

3. Second Chern number and its physical consequences

B. TRI topological insulators based on lattice Dirac models

- Diagonalized Hamiltonian in **k**-space

$$H = \sum_{\mathbf{k}} \psi_{\mathbf{k}}^{\dagger} \left[\sum_{i_0} \sin(k_{i_0}) \Gamma^{i_0} + \left(m + c \sum_{i_0} \cos(k_{i_0}) \right) \Gamma^0 \right] \psi_{\mathbf{k}}$$

- This Hamiltonian can be written in the compact form

$$H = \sum_{\mathbf{k}} \psi_{\mathbf{k}}^{\dagger} d_a(\mathbf{k}) \Gamma^a \psi_{\mathbf{k}} = \sum_{\mathbf{k}} \psi_{\mathbf{k}}^{\dagger} h(\mathbf{k}) \psi_{\mathbf{k}} \quad \Rightarrow \quad h(\mathbf{k}) = \sum_a d_a(\mathbf{k}) \Gamma^a = d(\mathbf{k}) \cdot \Gamma \qquad d(\mathbf{k}) = \begin{bmatrix} m + c\sum_i \cos(k_i) \\ \sin(k_x) \\ \sin(k_y) \\ \sin(k_z) \\ \sin(k_w) \end{bmatrix}$$

- The second Chern number can be written as

$$C_2 = -\frac{\pi^2}{15} \epsilon^{\mu\nu\rho\sigma\tau} \int \frac{d^4k\, d\omega}{(2\pi)^5} \mathrm{Tr} \left[\left(G \frac{\partial G^{-1}}{\partial q^\mu} \right) \left(G \frac{\partial G^{-1}}{\partial q^\nu} \right) \left(G \frac{\partial G^{-1}}{\partial q^\rho} \right) \left(G \frac{\partial G^{-1}}{\partial q^\sigma} \right) \left(G \frac{\partial G^{-1}}{\partial q^\tau} \right) \right]$$

$$G(q^\mu) = \frac{1}{\omega + i\delta - h(k_i)} \qquad q^\mu = (\omega, k_1, k_2, k_3, k_4)$$

- In terms of the compact Hamiltonian we get

$$C_2 = \frac{3}{8\pi^2} \int d^4\mathbf{k}\, \epsilon^{abcde} \hat{d}_a(\mathbf{k}) \frac{\partial \hat{d}_b(\mathbf{k})}{\partial k_x} \frac{\partial \hat{d}_c(\mathbf{k})}{\partial k_y} \frac{\partial \hat{d}_d(\mathbf{k})}{\partial k_z} \frac{\partial \hat{d}_e(\mathbf{k})}{\partial k_w}$$

- The critical values of *m* can be found as solutions to $\sum_a d_a^2(\mathbf{k}, m) = 0 \quad \Rightarrow \quad m = \begin{cases} -4c, & \mathbf{k} = (0,0,0,0) \\ -2c, & \mathbf{k} \in P\left[(\pi,0,0,0)\right] \\ 0, & \mathbf{k} \in P\left[(\pi,0,\pi,0)\right] \\ 2c, & \mathbf{k} \in P\left[(\pi,\pi,\pi,0)\right] \\ 4c, & \mathbf{k} = (\pi,\pi,\pi,\pi) \end{cases}$

- The function *P*[**k**] is a set of all the wavevectors obtained from index permutations of wavevector **k**. For example:

$$P[(\pi,0,0,0)] = \{(\pi,0,0,0),\ (0,\pi,0,0),\ (0,0,\pi,0),\ (0,0,0,\pi)\}$$

- The second Chern number C_2 for the different regions in parameter space, separated by critical values of *m*, can be evaluated approximately near the critical points to give:

$$C_2(m) = \begin{cases} 0, & m < -4c \\ 1, & -4c < m < -2c \\ -3, & -2c < m < 0 \\ 3, & 0 < m < 2c \\ -1, & 2c < m < 4c \\ 0 & m > 4c \end{cases}$$

4. Dimensional reduction to (3+1)-d TRI insulators

A. Effective action of (3+1)-d insulators

- Hamiltonian of Dirac model coupled to an external $U(1)$ gauge field

$$H[A] = \sum_{\mathbf{n},\mathbf{i}} \left[\psi_{\mathbf{n}}^{\dagger} \left(\frac{c\Gamma^0 - i\Gamma^i}{2}\right) e^{iA_{\mathbf{n},\mathbf{n}+\mathbf{i}}}\psi_{\mathbf{n}+\mathbf{i}} + \text{h.c.}\right] + m\sum_{\mathbf{n}} \psi_{\mathbf{n}}^{\dagger}\Gamma^0\psi_{\mathbf{n}}$$

- Consider a special "Landau"-gauge configuration satisfying: $A_{\mathbf{n},\mathbf{n}+\mathbf{i}} = A_{\mathbf{n}+\mathbf{w},\mathbf{n}+\mathbf{w}+\mathbf{i}}$
- We have translational invariance in the **w**-direction; $k_{\mathbf{w}}$ is a good quantum number. Hamiltonian can be rewritten as:

$$H[A] = \sum_{k_{\mathbf{w}},\mathbf{x},\mathbf{s}} \left[\psi_{\mathbf{x},k_{\mathbf{w}}}^{\dagger} \left(\frac{c\Gamma^0 - i\Gamma^s}{2}\right) e^{iA_{\mathbf{x},\mathbf{x}+\mathbf{s}}}\psi_{\mathbf{x}+\mathbf{s},k_{\mathbf{w}}} + \text{h.c.}\right] + \sum_{k_{\mathbf{w}},\mathbf{x},\mathbf{s}} \psi_{\mathbf{x},k_{\mathbf{w}}}^{\dagger} \left[\sin\left(k_{\mathbf{w}} + A_{\mathbf{x}4}\right)\Gamma^4 + \left(m + c\cos\left(k_{\mathbf{w}} + A_{\mathbf{x}4}\right)\right)\Gamma^0\right]\psi_{\mathbf{x},k_{\mathbf{w}}}$$

$$\mathbf{n} = (x,y,z,w),\ \mathbf{i} = \hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}},\hat{\mathbf{w}},\ A_{\mathbf{x}4} \equiv A_{\mathbf{x},\mathbf{x}+\mathbf{w}},\ \mathbf{x} = (x,y,z),\ \mathbf{s} = \hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}$$

- On a hyper-cylinder we can define: $\theta_{\mathbf{x}} \equiv k_{\mathbf{w}} + A_{\mathbf{x}4}$
- Since different $k_{\mathbf{w}}$ are decoupled we obtain the (3+1)-d model:

$$H_{3\text{D}}[A,\theta] = \sum_{\mathbf{x},\mathbf{s}} \left[\psi_{\mathbf{x}}^{\dagger} \left(\frac{c\Gamma^0 - i\Gamma^s}{2}\right) e^{iA_{\mathbf{x},\mathbf{x}+\mathbf{s}}}\psi_{\mathbf{x}+\mathbf{s}} + \text{h.c.}\right] + \sum_{\mathbf{x},\mathbf{s}} \psi_{\mathbf{x}}^{\dagger}\left[\sin(\theta_{\mathbf{x}})\Gamma^4 + \left(m + c\cos(\theta_{\mathbf{x}})\right)\Gamma^0\right]\psi_{\mathbf{x}}$$

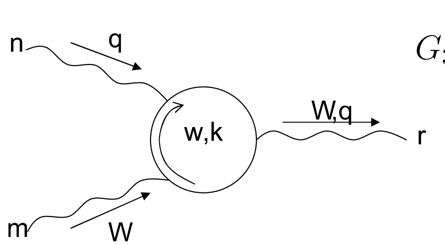
- To study the response properties of the (3+1)-d system, the effective action can be defined as:

$$\exp(iS_{3\text{D}}[A,\theta]) = \int D[\psi]D[\overline{\psi}]\exp\left(i\int dt\left[\sum_{\mathbf{x}}\overline{\psi}_{\mathbf{x}}\left(i\partial_\tau - A_{\mathbf{x}0}\right)\psi_{\mathbf{x}} - H[A,\theta]\right]\right).$$

- We can Taylor expand around the field configuration: $A_s(\mathbf{x},t) \equiv 0,\ \theta(\mathbf{x},t) \equiv \theta_0$

$$S_{3\text{D}} = \frac{G_3(\theta_0)}{4\pi}\int d^3\mathbf{x}\ dt\ \epsilon^{\mu\nu\sigma\tau}\delta\theta(\mathbf{x},t)\ \partial_\mu A_\nu \partial_\sigma A_\tau \qquad \delta\theta(\mathbf{x},t) = \theta(\mathbf{x},t) - \theta_0$$

- The coefficient $G_3(\theta_0)$ is determined by the below Feynman diagram



$$G_3(\theta_0) = -\frac{\pi}{6}\int \frac{d^3k\ d\omega}{(2\pi)^4}\text{Tr}\left\{\epsilon^{\mu\nu\sigma\tau}\left[\left(G\frac{\partial G^{-1}}{\partial q^\mu}\right)\left(G\frac{\partial G^{-1}}{\partial q^\nu}\right)\left(G\frac{\partial G^{-1}}{\partial q^\sigma}\right)\left(G\frac{\partial G^{-1}}{\partial q^\tau}\right)\left(G\frac{\partial G^{-1}}{\partial \theta_0}\right)\right]\right\}$$

$$q^\mu = (\omega, k_x, k_y, k_z)$$

4. Dimensional reduction to (3+1)-d TRI insulators

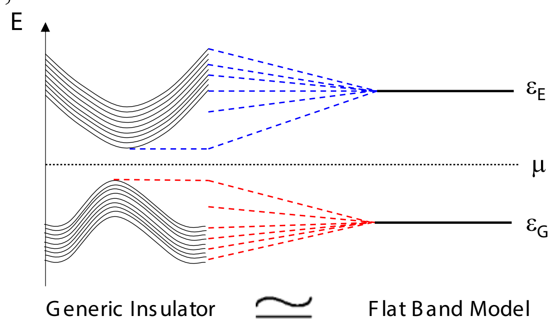
A. Effective action of (3+1)-d insulators

- Copying coefficient $G_3(\theta_0)$ from last slide

$$G_3(\theta_0) = -\frac{\pi}{6}\int \frac{d^3k\ d\omega}{(2\pi)^4}\mathrm{Tr}\left\{\epsilon^{\mu\nu\sigma\tau}\left[\left(G\frac{\partial G^{-1}}{\partial q^\mu}\right)\left(G\frac{\partial G^{-1}}{\partial q^\nu}\right)\left(G\frac{\partial G^{-1}}{\partial q^\sigma}\right)\left(G\frac{\partial G^{-1}}{\partial q^\tau}\right)\left(G\frac{\partial G^{-1}}{\partial \theta_0}\right)\right]\right\}$$

- If we define the 4-D Berry connection can be defined as: $f_{ij}^{\alpha\beta} = \partial_i a_j^{\alpha\beta} - \partial_j a_i^{\alpha\beta} + i\left[a_i, a_j\right]^{\alpha\beta}$ $\quad k_\mu \equiv (k_x, k_y, k_z, \theta_0)$
- Then we can show that the above expression for $G_3(\theta_0)$, in terms of Green functions, reduces to:

$$G_3(\theta_0) = \frac{1}{8\pi^2}\int d^3\mathbf{k}\ \epsilon^{ijk}\mathrm{tr}\left[f_{\theta i} f_{jk}\right]$$



- The equivalence of these two forms can be proved in **three** steps:
  - Derive an expression $h(\mathbf{k}, t)$ showing adiabatic connection between the expression for arbitrary $h(\mathbf{k})$ and maximally degenerate $h_0(\mathbf{k})$
  - Explicit evaluation of $G_3(\theta_0)$ using Green functions for $h_0(\mathbf{k})$
  - Topological invariance of $G_3(\theta_0)$ under adiabatic deformations of $h(\mathbf{k})$
- **Step I**: Adiabatic connection of $h(\mathbf{k})$ to $h_0(\mathbf{k})$
  - Any single particle Hamiltonian $h(\mathbf{k})$ can be diagonalized as $h(\mathbf{k}) = U(\mathbf{k})D(\mathbf{k})U^\dagger(\mathbf{k})$ where $U(\mathbf{k}) = (\left|1,\mathbf{k}\right\rangle, \left|2,\mathbf{k}\right\rangle, ..., \left|N,\mathbf{k}\right\rangle)$ and $D(\mathbf{k}) = \mathrm{diag}\left[\epsilon_1(\mathbf{k}), \epsilon_2(\mathbf{k}), ..., \epsilon_N(\mathbf{k})\right]$
  - With zero chemical potential, for an insulator with M full bands, the eigenvalues satisfy $\epsilon_1(\mathbf{k}) \le \epsilon_2(\mathbf{k}) \le ... \le \epsilon_M(\mathbf{k}) < 0 \le \epsilon_{M+1}(\mathbf{k}) \le ... \le \epsilon_N(\mathbf{k})$.
  - For $t \in [0,1]$ we can define the interpolation
  $$E_\alpha(\mathbf{k}, t) = \begin{cases}\epsilon_\alpha(\mathbf{k})(1-t) + \epsilon_G t, & 1 \le \alpha \le M\\ \epsilon_\alpha(\mathbf{k})(1-t) + \epsilon_E t, & M < \alpha \le N\end{cases}$$
  and $D_0(\mathbf{k}, t) = \mathrm{diag}\left[E_1(\mathbf{k}, t), E_2(\mathbf{k}, t), ..., E_N(\mathbf{k}, t)\right]$
  - Then we get
  $$D_0(\mathbf{k}, 0) = D(\mathbf{k}),\ \ D_0(\mathbf{k}, 1) = \begin{pmatrix}\epsilon_G \mathbb{I}_{M\times M} & \\ & \epsilon_E \mathbb{I}_{N-M\times N-M}\end{pmatrix}$$
  - We obtain an adiabatic interpolation between $h(\mathbf{k}, 0) = h(\mathbf{k})$ and $h(\mathbf{k}, 1) = U(\mathbf{k})D_0(\mathbf{k}, 1)U^\dagger(\mathbf{k}) = h_0(\mathbf{k})$
  $$h_0(\mathbf{k}) = \epsilon_G \sum_{\alpha=1}^{M}\left|\alpha,\mathbf{k}\right\rangle\left\langle\alpha,\mathbf{k}\right| + \epsilon_E \sum_{\beta=M+1}^{N}\left|\beta,\mathbf{k}\right\rangle\left\langle\beta,\mathbf{k}\right| = \epsilon_G P_G(\mathbf{k}) + \epsilon_E P_E(\mathbf{k})$$

4. Dimensional reduction to (3+1)-d TRI insulators

A. Effective action of (3+1)-d insulators

- **Step II**: Explicit evaluation of $G_3(\theta_0)$ using Green's function for $h_0(\mathbf{k})$
  - The expression for $h_0(\mathbf{k})$ from step I:

$$h_0(\mathbf{k}) = \epsilon_G \sum_{\alpha=1}^{M} |\alpha,\mathbf{k}\rangle\langle\alpha,\mathbf{k}| + \epsilon_E \sum_{\beta=M+1}^{N} |\beta,\mathbf{k}\rangle\langle\beta,\mathbf{k}| = \epsilon_G P_G(\mathbf{k}) + \epsilon_E P_E(\mathbf{k})$$

  - Properties of "projection" operators $P_G(\mathbf{k})$ and $P_E(\mathbf{k})$:
    $P_G^2 = P_G,\ P_E^2 = P_E,\ P_E P_G = P_G P_E = 0$
  - The Green's function for $h_0(\mathbf{k})$:

$$G(\mathbf{k},\omega) = \frac{1}{\omega + i\delta - \epsilon_G P_G(\mathbf{k}) - \epsilon_E P_E(\mathbf{k})} = c_1 P_G(\mathbf{k}) + c_2 P_E(\mathbf{k})$$

$$\begin{aligned} 1 &= (\omega + i\delta - \epsilon_G P_G(\mathbf{k}) - \epsilon_E P_E(\mathbf{k}))\,(c_1 P_G(\mathbf{k}) + c_2 P_E(\mathbf{k})) \\ &= \big((\omega + i\delta)\,(c_1 P_G(\mathbf{k}) + c_2 P_E(\mathbf{k})) - \big(c_1\epsilon_G P_G^2(\mathbf{k}) + c_2\epsilon_G P_G(\mathbf{k})P_E(\mathbf{k})\big) - \big(c_1\epsilon_E P_E(\mathbf{k})P_G(\mathbf{k}) + c_2\epsilon_E P_E^2(\mathbf{k})\big)\big) \\ &= ((\omega + i\delta)\,(c_1 P_G(\mathbf{k}) + c_2 P_E(\mathbf{k})) - (c_1\epsilon_G P_G(\mathbf{k}) + 0) - (0 + c_2\epsilon_E P_E(\mathbf{k}))) \\ &= c_1\,(\omega + i\delta - \epsilon_G)\,P_G(\mathbf{k}) + c_2\,(\omega + i\delta - \epsilon_E)\,P_E(\mathbf{k}) \end{aligned}$$

$$P_G(\mathbf{k}) = c_1\,(\omega + i\delta - \epsilon_G)\,P_G^2(\mathbf{k}) + c_2\,(\omega + i\delta - \epsilon_E)\,P_E(\mathbf{k})P_G(\mathbf{k})$$

$$P_G(\mathbf{k}) = c_1\,(\omega + i\delta - \epsilon_G)\,P_G(\mathbf{k})$$

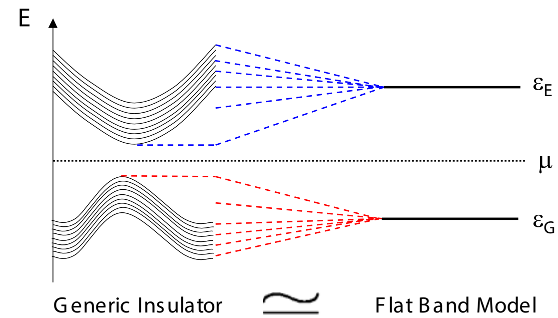
$$\boxed{c_1 = \frac{1}{\omega + i\delta - \epsilon_G} \qquad c_2 = \frac{1}{\omega + i\delta - \epsilon_E}}$$

$$\boxed{\begin{aligned} P_G^2 &= \sum_{\alpha=1}^{M} |\alpha,\mathbf{k}\rangle\langle\alpha,\mathbf{k}| \sum_{\alpha'=1}^{M} |\alpha',\mathbf{k}\rangle\langle\alpha',\mathbf{k}| \\ &= \sum_{\alpha=1}^{M}\sum_{\alpha'=1}^{M} |\alpha,\mathbf{k}\rangle\langle\alpha,\mathbf{k}|\alpha',\mathbf{k}\rangle\langle\alpha',\mathbf{k}| \\ &= \sum_{\alpha=1}^{M}\sum_{\alpha'=1}^{M} |\alpha,\mathbf{k}\rangle\,\delta_{\alpha\alpha'}\,\langle\alpha',\mathbf{k}| \\ &= \sum_{\alpha=1}^{M} |\alpha,\mathbf{k}\rangle\langle\alpha,\mathbf{k}| = P_G \end{aligned}}$$

- Consider the maximally degenerate or flat-band Hamiltonian

$$h_0(\mathbf{k}) = \epsilon_G P_G(\mathbf{k}) + \epsilon_E P_E(\mathbf{k})$$

$$\frac{\partial h_0(\mathbf{k})}{\partial k_i} = \epsilon_G \frac{\partial P_G(\mathbf{k})}{\partial k_i} + \epsilon_E \frac{\partial P_E(\mathbf{k})}{\partial k_i} = 0 \quad \Rightarrow \quad \boxed{\epsilon_G \frac{\partial P_G(\mathbf{k})}{\partial k_i} = -\epsilon_E \frac{\partial P_E(\mathbf{k})}{\partial k_i}}$$



- Using the Green's function we can write

$$\frac{\partial G^{-1}(\mathbf{k},\omega)}{\partial\omega} = 1, \qquad \frac{\partial G^{-1}(\mathbf{k},\omega)}{\partial k_i} = -\epsilon_G \frac{\partial P_G(\mathbf{k})}{\partial k_i} - \epsilon_E \frac{\partial P_E(\mathbf{k})}{\partial k_i} = (\epsilon_E - \epsilon_G)\,\frac{\partial P_G(\mathbf{k})}{\partial k_i}$$

4. Dimensional reduction to (3+1)-d TRI insulators

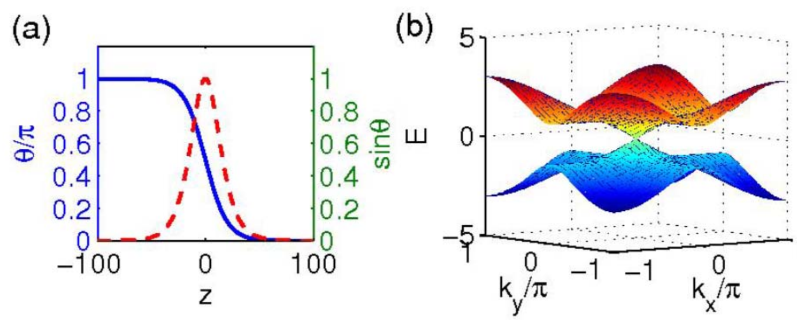
A. Effective action of (3+1)-d insulators

- **Step II**: Explicit evaluation of $G_3(\theta_0)$ using Green's function for $h_0(\mathbf{k})$
  - Properties of "projection" operators $P_G(\mathbf{k})$ and $P_E(\mathbf{k})$:
    $P_G^2 = P_G,\ P_E^2 = P_E,\ P_E P_G = P_G P_E = 0$
  - The Green's function for $h_0(\mathbf{k})$:
    $$G(\mathbf{k},\omega) = \frac{P_G(\mathbf{k})}{\omega + i\delta - \epsilon_G} + \frac{P_E(\mathbf{k})}{\omega + i\delta - \epsilon_E} \;\Rightarrow\; \frac{\partial G^{-1}(\mathbf{k},\omega)}{\partial \omega} = 1, \quad \frac{\partial G^{-1}(\mathbf{k},\omega)}{\partial k_i} = (\epsilon_E - \epsilon_G)\frac{\partial P_G(\mathbf{k})}{\partial k_i}$$
  - Recall the expression for $G_3(\theta_0)$
    $$G_3(\theta_0) = -\frac{\pi}{6}\int \frac{d^3k\ d\omega}{(2\pi)^4}\mathrm{Tr}\left\{\epsilon^{\mu\nu\sigma\tau}\left[\left(G\frac{\partial G^{-1}}{\partial q^\mu}\right)\left(G\frac{\partial G^{-1}}{\partial q^\nu}\right)\left(G\frac{\partial G^{-1}}{\partial q^\sigma}\right)\left(G\frac{\partial G^{-1}}{\partial q^\tau}\right)\left(G\frac{\partial G^{-1}}{\partial \theta_0}\right)\right]\right\}$$
    $$C_2 = \int G_3(\theta_0) d\theta_0$$
  - Recall C_2 as well
    $$C_2 = -\frac{\pi^2}{15}\epsilon^{\mu\nu\rho\sigma\tau}\int \frac{d^4k\ d\omega}{(2\pi)^5}\mathrm{Tr}\left[\left(G\frac{\partial G^{-1}}{\partial q^\mu}\right)\left(G\frac{\partial G^{-1}}{\partial q^\nu}\right)\left(G\frac{\partial G^{-1}}{\partial q^\rho}\right)\left(G\frac{\partial G^{-1}}{\partial q^\sigma}\right)\left(G\frac{\partial G^{-1}}{\partial q^\tau}\right)\right]$$
  - Plugging in the above expressions for the Green's functions we get
    $$C_2 = -\frac{\pi^2}{3}\epsilon^{ijk\ell}\int \frac{d^4\mathbf{k}\ d\omega}{(2\pi)^5}\sum_{n,m,s,t=1,2}\frac{\mathrm{Tr}\left[P_n \frac{\partial P_G}{\partial k_i} P_m \frac{\partial P_G}{\partial k_j} P_s \frac{\partial P_G}{\partial k_k} P_t \frac{\partial P_G}{\partial k_\ell}\right](\epsilon_E - \epsilon_G)^4}{(\omega + i\delta - \epsilon_n)^2(\omega + i\delta - \epsilon_m)(\omega + i\delta - \epsilon_s)(\omega + i\delta - \epsilon_t)}$$
    where $P_{1/2}(\mathbf{k}) = P_{G/E}(\mathbf{k})$
    $$C_2 = \frac{1}{32\pi^2}\int d^4\mathbf{k}\ \epsilon^{ijk\ell}\mathrm{Tr}\left[f_{ij}f_{k\ell}\right]$$
  - Now, introduce the non-Abelian Chern-Simons term $\mathcal{K}^A = \frac{1}{16\pi^2}\epsilon^{ABCD}\mathrm{Tr}\left[\left(f_{BC} - \frac{1}{3}[a_B, a_C]\right)\cdot a_D\right]$
  - Analogy to the charge polarization defined as the integral of the adiabatic connection 1-form over a path in momentum space
    $$P_3(\theta_0) = \frac{1}{16\pi^2}\int d^3\mathbf{k}\ \epsilon^{\theta ijk}\mathrm{Tr}\left[\left(f_{ij} - \frac{1}{3}[a_i, a_j]\right)\cdot a_k\right]$$

4. Dimensional reduction to (3+1)-d TRI insulators

D. Physical properties of Z_2-nontrivial insulators

- The non-trivial insulator has a magnetoelectric polarization P_3 = 1/2 mod 1; the effective action of the bulk system should be
$$S_{3D} = \frac{2n+1}{8\pi}\int d^3\mathbf{x}\, dt\, \epsilon^{\mu\nu\sigma\tau}\partial_\mu A_\nu \partial_\sigma A_\tau$$
where $n = P_3 - 1/2 \in \mathbb{Z}$. S_{3D} is quantized (and independent of P_3) for compact space-time; hence P_3 is unphysical
- For open boundaries consider semi-infinite nontrivial insulator occupying the space $z \leq 0$; the rest ($z > 0$) is vacuum
- Effective action for all of $\mathbb{R}^3$ as
$$S_{3D} = \frac{1}{4\pi}\int d^3\mathbf{x}\, dt\, \epsilon^{\mu\nu\sigma\tau} A_\mu \partial_\nu P_3\, \partial_\sigma A_\tau$$
- Since P_3 = 1/2 mod 1 for $z < 0$ and P_3 = 0 mod 1 for $z > 0$, we have $\partial_z P_3 = \left(n + \frac{1}{2}\right)\delta(z)$
- The domain wall of P_3 carries a quantum Hall effect
- The Hall conductance carried by a Dirac fermion is well-defined only when the fermion mass is non-vanishing, so that a gap is opened
$$H = k_x\sigma^x + k_y\sigma^y + m\sigma^z$$
- If the surface (initially) has $2n$+1 Dirac cones. Breaking time-reversal symmetry on the surface gives the following Hall conductance from *each* Dirac cone
$$\sigma_H = \frac{1}{4\pi}\text{sgn}(m)\quad\left(= \frac{e^2}{2h}\text{sgn}(m)\right)$$



- We can explicitly see this in the model: $\theta(\mathbf{x}) = \theta(z) = \frac{\pi}{2}\left[1 - \tanh(z/4\xi)\right]$
- Then the Dirac Hamiltonian looks like:

$$H = \sum_{z,k_x,k_y}\left[\psi^\dagger_{k_xk_y}(z)\left(\frac{c\Gamma^0 - i\Gamma^3}{2}\right)\psi_{k_xk_y}(z+1) + \text{h.c.}\right] + \sum_{z,k_x,k_y}\psi^\dagger_{k_xk_y}(z)\left[(m + c\cos(\theta(z)) + c\cos(k_x) + c\cos(k_y)) + \sin(k_x)\Gamma^1 + \sin(k_y)\Gamma^2\right]\psi_{k_xk_y}(z) \quad \leftarrow H_0$$
$$+ \sum_{z,k_x,k_y}\psi^\dagger_{k_xk_y}(z)\sin(\theta(z))\Gamma^4\psi_{k_xk_y}(z) \quad \leftarrow H_1$$

4. Dimensional reduction to (3+1)-d TRI insulators

D. Physical properties of Z_2-nontrivial insulators

- Recall Dirac Hamiltonian from last slide

$$H = \underbrace{\sum_{z,k_x,k_y}\left[\psi^\dagger_{k_xk_y}(z)\left(\frac{c\Gamma^0 - i\Gamma^3}{2}\right)\psi_{k_xk_y}(z+1) + \text{h.c.}\right] + \sum_{z,k_x,k_y}\psi^\dagger_{k_xk_y}(z)\left[(m + c\cos(\theta(z)) + c\cos(k_x) + c\cos(k_y)) + \sin(k_x)\Gamma^1 + \sin(k_y)\Gamma^2\right]\psi_{k_xk_y}(z)}_{H_0} + \underbrace{\sum_{z,k_x,k_y}\psi^\dagger_{k_xk_y}(z)\sin(\theta(z))\Gamma^4\psi_{k_xk_y}(z)}_{H_1}$$

- Under a time-reversal transformation, Γ^0 is even and $\Gamma^{1,2,3,4}$ are odd; in other words only H_0 respects time-reversal symmetry
- Recall second Chern number for this model

$$C_2(m) = \begin{cases} 0, & m < -4c \\ 1, & -4c < m < -2c \\ -3, & -2c < m < 0 \\ 3, & 0 < m < 2c \\ -1, & 2c < m < 4c \\ 0 & m > 4c \end{cases}$$

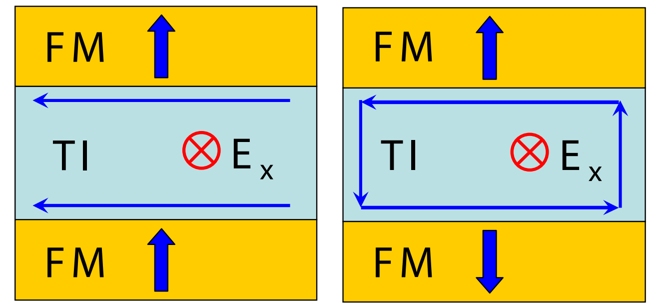
- For $\theta = \pi$ and $-4c < m < -2c$ we have $C_2 = 1$. It can be shown that for $\theta = 0$, the region $-4c < m < -2c$ is adiabatically connected to $m \rightarrow -\infty$ such that $C_2 = 0$.
- It can be noted that $\{\Gamma^4, H_0\} = 0$ in the bulk and on the surface
- The surface Hamiltonian of H_0 can be written as $H = k_x\sigma^x + k_y\sigma^y$
- Since the only thing that anticommutes with this above surface Hamiltonian, the surface Hamiltonian for $H = H_0 + H_1$ is $H = k_x\sigma^x + k_y\sigma^y + m\sigma^z$
- In summary, the effect of a time-reversal symmetry breaking term on the surface is to assign a mass to the Dirac fermions which determines the winding direction of P_3 through the domain wall
- An analogous Chern number, that can evaluate the winding, can be defined as $g[M(\mathbf{x})]$, where $M(\mathbf{x})$ is the T-breaking field. The surface action, in terms of $g[M(\mathbf{x})]$, can be written as

$$S_{\text{surf}} = \frac{1}{4\pi}\int_{\partial\mathcal{V}} d\hat{\mathbf{n}}_\mu \left(g[M(\mathbf{x})] + \frac{1}{2}\right)\epsilon^{\mu\nu\sigma\tau}A_\nu\partial_\sigma A_\tau$$

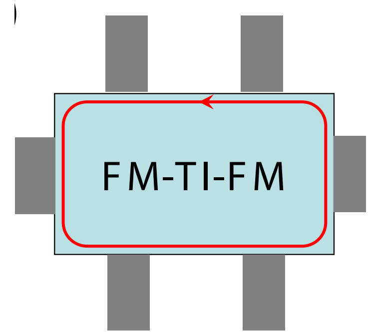
4. Dimensional reduction to (3+1)-d TRI insulators

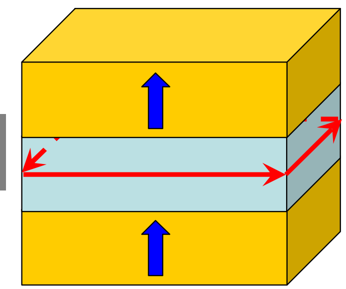
D. Physical properties of Z_2-nontrivial insulators

1. Magnetization-induced QH effect



- Consider a ferromagnet-topological insulator heterostructure
- The pointing of the magnetization is given by $\hat{\mathbf{M}}$
- For the first (parallel) case $\hat{\mathbf{n}}_t = \hat{\mathbf{M}} \quad \hat{\mathbf{n}}_b = -\hat{\mathbf{M}}$
- For the second (antiparallel) case $\hat{\mathbf{n}}_t = \hat{\mathbf{M}} \quad \hat{\mathbf{n}}_b = \hat{\mathbf{M}}$
- Then for $\mathbf{E} = E_x \hat{\mathbf{x}} \quad \mathbf{j} = \hat{\mathbf{M}} \times \mathbf{E}/4\pi$
- Using this expression the currents on the top and bottom surfaces can be determined and are indicated for the two cases by (horizontal) blue arrows
- The antiparallel magnetization leads to a vanishing net Hall conductance, while the parallel magnetization leads to $\sigma_H = e^2/h$
- A top-down view of the device, to measure this quantum Hall effect, can be seen in the bottom left figure
- For the case of parallel magnetization, the trapping of the chiral edge states on the side surfaces of the topological insulator can be seen in the 3D view in the bottom right figure; these carry the quantized Hall current.



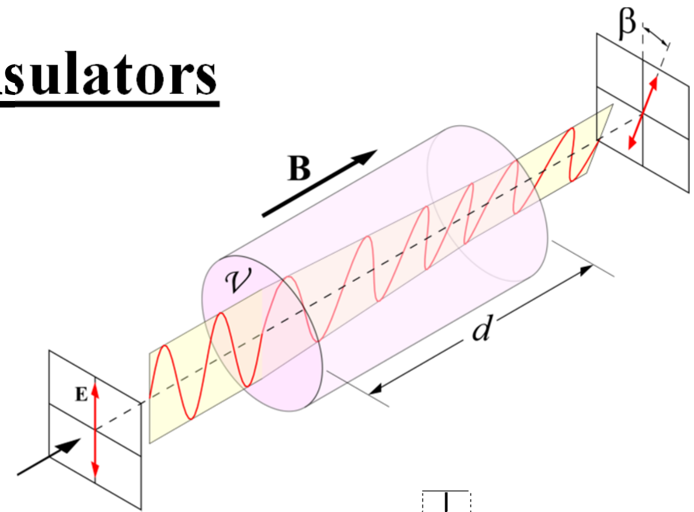


- Recall, in a T-breaking field $M(\mathbf{x})$, the surface Chern-Simons action: $S_{\text{surf}} = \frac{1}{4\pi} \int_{\partial\mathcal{V}} d\hat{\mathbf{n}}_\mu \left(g[M(\mathbf{x})] + \frac{1}{2} \right) \epsilon^{\mu\nu\sigma\tau} A_\nu \partial_\sigma A_\tau$
- **Note**: Although the Hall conductance of the two surfaces are the same in the global x, y, z basis, they are opposite in the local basis defined with respect to the normal vector
- This corresponds to $g[M(\mathbf{x})] + 1/2 = \pm 1/2$ for the top and bottom surfaces respectively.
- Drawing closer analogy to the integer quantum Hall effect (IQHE), we could alternatively say $g[M(\mathbf{x})] = 0$ (-1) for the top (bottom) surface; i.e. there exists a *domain wall* between two adjacent plateaux of IQHE
- Such a domain wall will trap a chiral Fermi liquid which is responsible for the net Hall effect
- **Note**: in general there will also be other non-chiral states on this side surface; they are irrelevant since only one chiral edge state is protected
- Conditions for observing this type of quantum Hall: (i) $k_B T \ll E_M$ (magnetization-induced *bulk* gap), (ii) chemical potential lies in the bulk gap
- **Question**: is this special type of quantum Hall a topological phase or symmetry-protected topological phase?

4. Dimensional reduction to (3+1)-d TRI insulators

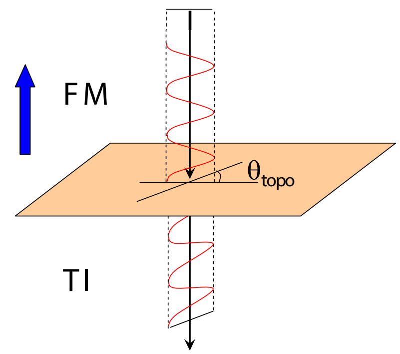
D. Physical properties of Z_2-nontrivial insulators

3. Low-frequency Faraday rotation



- It is the rotation of polarization of light by a certain angle $\beta = B\mathcal{V}d$
- We can tune polarization by tuning the applied magnetic field
- Perhaps the same can be accomplished with external electric fields by exploiting TME in a medium with nonzero θ
- Electric fields are, in general, easier to generate than magnetic fields
- Recall that the total action of the electromagnetic field including the topological term is given by

$$S_{\rm tot} = \int d^3\mathbf{x}\, dt \left[\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}F_{\mu\nu}\mathcal{P}^{\mu\nu} - \frac{1}{c}j^{\mu}A_{\mu}\right] + \frac{\alpha}{16\pi}\int d^3\mathbf{x}\, dt\ P_3\ \epsilon^{\mu\nu\sigma\tau}F_{\mu\nu}F_{\sigma\tau}$$



- However, the effective theory only applies in the low-energy limit $E \ll E_g$, where E_g is the gap of the surface state
- Assuming a FM-TI interface at $z = 0$, normally incident, linearly-polarized light in either region can be written as (primed variables belong to the TI):

$$\mathbf{A}(z,t) = \begin{cases} \mathbf{a}e^{i(-kz-\omega t)} + \mathbf{b}e^{i(kz-\omega t)}, & z > 0 \\ \mathbf{c}e^{i(-k'z-\omega t)}, & z < 0 \end{cases}$$

- The ΔP_3 terms in the action contribute unconventional boundary conditions at $z = 0$ given by ($\mathbf{a} = a_x + ia_y$, etc.)

$$\mathbf{a} + \mathbf{b} = \mathbf{c}$$
$$\hat{\mathbf{z}} \times [k(-\mathbf{a}+\mathbf{b})/\mu + k'\mathbf{c}/\mu'] = -\frac{2\alpha\Delta\omega}{c}\mathbf{c}$$

- Solving the above equations simultaneously we get: $a_+ = \frac{1}{2}\left[1 + \frac{k'/\mu' - 2i\alpha\Delta\omega/c}{k/\mu}\right]c_+$
- By simple algebra the polarization plane rotated by an angle is given by

$$\theta_{\rm topo} = \arctan\left\{\frac{2\alpha\Delta}{\sqrt{\epsilon/\mu} + \sqrt{\epsilon'/\mu'}}\right\}$$

- For typical values of $E_g = 10$ meV (and $\epsilon, \mu \sim 1$ and $\Delta = 1/2$) we get EM frequency $\ll$ 2.4 THz, which is in the far infrared or microwave region
- In principle, it is possible to find a topological insulator with a larger gap which can support an accurate measurement of Faraday rotation