Introduction to topological superconductivity and Majorana fermions

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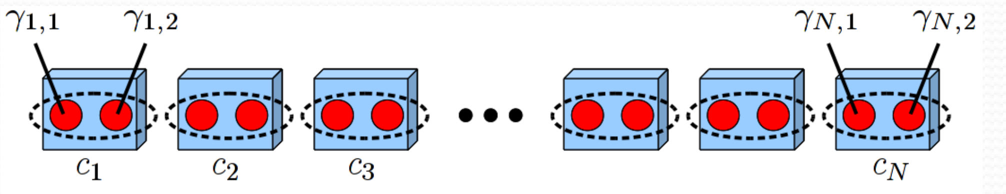
Outline

- **Introduction**
- **Majorana fermions in *p*-wave superconductors**
  - Representation in terms of fermionic operators
  - Non-abelian statistics
  - Majorana qubits and topological quantum computation
- **Proximity-induced superconductivity in spin-orbit semiconductors**
- **Induced *p*-wave-like gap in semiconductors**
- **Conclusions and outlook**

Kitaev 1-D Chain



- **Spinless *p*-wave superconductor**
  - Tight-Binding Hamiltonian

$$\mathcal{H}_{\text{chain}} = -\mu \sum_{i=1}^{N} n_i - \sum_{i=1}^{N-1} \left(t c_i^\dagger c_{i+1} + \Delta c_i c_{i+1} + \text{h.c.} \right)$$

  - Defining Majorana Operators

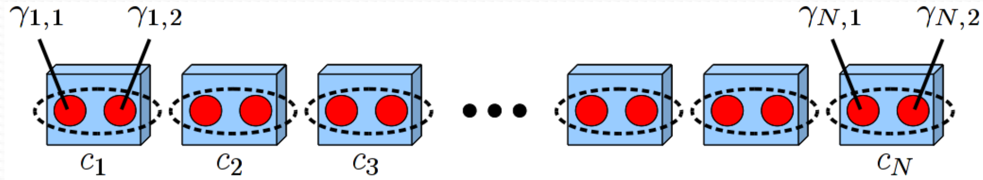
$$c_i = \frac{1}{2}(\gamma_{i,2} + i\gamma_{i,1}) \quad c_i^\dagger = \frac{1}{2}(\gamma_{i,2} - i\gamma_{i,1}) \quad \Rightarrow \quad \gamma_{i,2} = c_i^\dagger + c_i \quad \gamma_{i,1} = i\left(c_i^\dagger - c_i\right)$$

- **Anticommutation relations for Majorana Operators**
  - Special case: “left” Majoranas on different sites

$$\begin{aligned} \{\gamma_{i,1}, \gamma_{j,1}\} &= \{c_i^\dagger, c_j^\dagger\} + \{c_i^\dagger, c_j\} + \{c_i, c_j^\dagger\} + \{c_i, c_j\} \\ &= 0 + \delta_{ij} + \delta_{ij} + 0 \\ &= 2\delta_{ij} \end{aligned}$$

  - Majoranas on same site: $\{\gamma_{i,k}, \gamma_{i,\ell}\} = 2\delta_{k\ell} \quad \Rightarrow \quad \boxed{\{\gamma_{i,k}, \gamma_{j,\ell}\} = 2\delta_{ij}\delta_{k\ell}}$

Kitaev 1-D Chain



- **Hamiltonian in terms of Majorana Operators**
  - Simple case $\mu = 0,\ t = \Delta$

$$\mathcal{H}_{\text{chain}} = -t\sum_{i=1}^{N-1}\left(c_i^\dagger c_{i+1} + c_i c_{i+1} + c_{i+1}^\dagger c_i + c_{i+1}^\dagger c_i^\dagger\right)$$

$$\boxed{\gamma_{i,2} = c_i^\dagger + c_i}$$

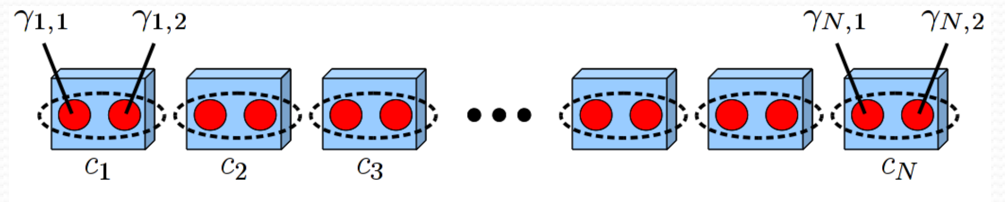
$$= -t\sum_{i=1}^{N-1}\left(\left(c_i^\dagger + c_i\right)c_{i+1} + c_{i+1}^\dagger\left(c_i + c_i^\dagger\right)\right) = -t\sum_{i=1}^{N-1}\left(\gamma_{i,2}c_{i+1} + c_{i+1}^\dagger\gamma_{i,2}\right)$$

$$= -\frac{t}{2}\sum_{i=1}^{N-1}(\gamma_{i,2}\gamma_{i+1,2} + i\gamma_{i,2}\gamma_{i+1,1} + \gamma_{i+1,2}\gamma_{i,2} - i\gamma_{i+1,1}\gamma_{i,2})$$

  - Recall anticommutation $\{\gamma_{i,k}, \gamma_{j,\ell}\} = 2\delta_{ij}\delta_{k\ell}$

$$\mathcal{H}_{\text{chain}} = -\frac{t}{2}\sum_{i=1}^{N-1}(\gamma_{i,2}\gamma_{i+1,2} + i\gamma_{i,2}\gamma_{i+1,1} - \gamma_{i,2}\gamma_{i+1,2} + i\gamma_{i,2}\gamma_{i+1,1}) = \boxed{-it\sum_{i=1}^{N-1}\gamma_{i,2}\gamma_{i+1,1}}$$

Kitaev 1-D Chain

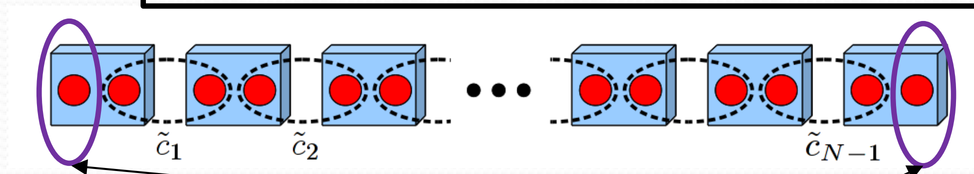


- **Alternative pairing of Majorana fermions**
  - Recall 1-D Majorana Hamiltonian

$$\mathcal{H}_{\text{chain}} = -it \sum_{i=1}^{N-1} \gamma_{i,2}\gamma_{i+1,1}$$

$$\boxed{c_i = \frac{1}{2}(\gamma_{i,2} + i\gamma_{i,1}) \quad c_i^\dagger = \frac{1}{2}(\gamma_{i,2} - i\gamma_{i,1})}$$

  - Define $\tilde{c}_i = \frac{1}{2}(\gamma_{i+1,1} + i\gamma_{i,2})$



$$\tilde{c}_N = (\gamma_{N,2} + i\gamma_{1,1})/2$$

$$\tilde{c}_i^\dagger \tilde{c}_i = \frac{1}{4}(\gamma_{i+1,1} - i\gamma_{i,2})(\gamma_{i+1,1} + i\gamma_{i,2})$$

$$= \frac{1}{4}(\gamma_{i+1,1}^2 + i\gamma_{i+1,1}\gamma_{i,2} - i\gamma_{i,2}\gamma_{i+1,1} - i^2\gamma_{i,2}^2) \qquad \boxed{\{\gamma_i, \gamma_j\} = 2\delta_{ij}}$$

$$= \frac{1}{4}(1 - i\gamma_{i,2}\gamma_{i+1,1} - i\gamma_{i,2}\gamma_{i+1,1} + 1) = \frac{1}{2}(1 - i\gamma_{i,2}\gamma_{i+1,1}) \Rightarrow -i\gamma_{i,2}\gamma_{i+1,1} = 2\tilde{c}_i^\dagger\tilde{c}_i - 1$$

  - Diagonalized Hamiltonian $\mathcal{H}_{\text{chain}} = 2t\sum_{i=1}^{N-1} \tilde{c}_i^\dagger \tilde{c}_i - \cancel{(N-1)t}$

Role of pairing in Kitaev 1-D Chain

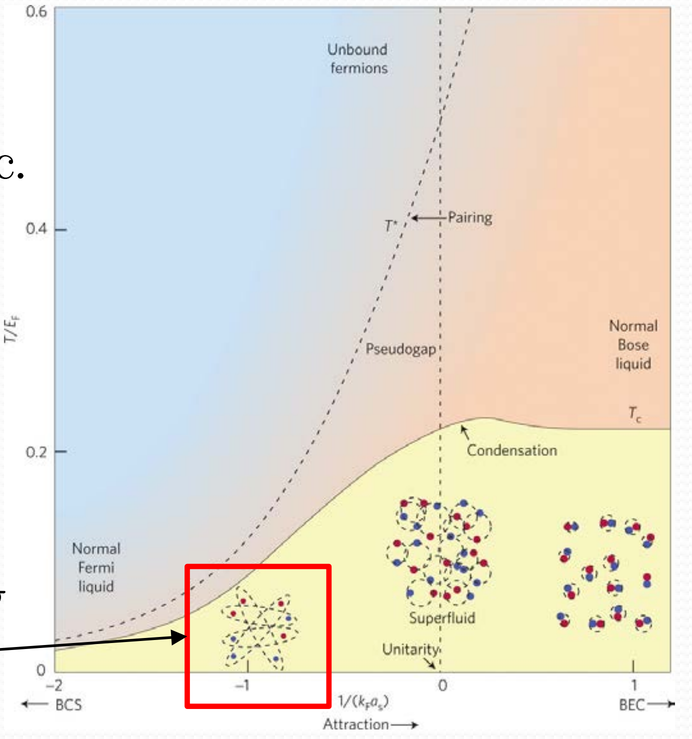
- **What is the nature of pairing?**
  - Recall the tight-Binding Hamiltonian

$$\mathcal{H}_{\text{chain}} = -\mu \sum_{i=1}^{N} n_i - \sum_{i=1}^{N-1} \left(t c_i^\dagger c_{i+1} + \Delta \boxed{c_i c_{i+1}} + \text{h.c.} \right)$$

  - Why is this a *p*-wave superconductor?
  - For the so-called *s*-, *p*-, *d*- or *f*-wave superconductor $\Delta(\mathbf{k}) = \text{constant},\ k_x + i k_y,\ \text{etc.}$
- **Pairing in real space**
  - How to visualize cooper pairs?
  - Lattice model for (conventional) *s*-wave

$$\mathcal{H} = -t \sum_{\langle i,j \rangle \sigma} c_{i\sigma}^\dagger c_{j\sigma} - \mu \sum_{i\sigma} n_{i\sigma} - U \sum_i n_{i\uparrow} n_{i\downarrow}$$

  - On-site particle number operator $n_{i\sigma} = c_{i\sigma}^\dagger c_{i\sigma}$
  - This “bosonic blob” still at the same site



Role of pairing in Kitaev 1-D Chain

- **Pairing in "conventional" superconductivity**
  - Recall lattice model for conventional superconductor

$$\mathcal{H} = -t\sum_{\langle i,j\rangle\sigma} c^\dagger_{i\sigma}c_{j\sigma} - \boxed{\mu\sum_{i\sigma} n_{i\sigma}} - U\sum_i n_{i\uparrow}n_{i\downarrow}$$

$\mu \to \mu'$

  - Applying Wick's theorem

$$\langle n_{i\uparrow}n_{i\downarrow}\rangle = \langle c^\dagger_{i\uparrow}c_{i\uparrow}c^\dagger_{i\downarrow}c_{i\downarrow}\rangle = \boxed{\langle c^\dagger_{i\uparrow}c_{i\uparrow}\rangle\langle c^\dagger_{i\downarrow}c_{i\downarrow}\rangle} + \underbrace{\boxed{\langle c^\dagger_{i\uparrow}c^\dagger_{i\downarrow}\rangle\langle c_{i\uparrow}c_{i\downarrow}\rangle}}_{\Delta^*\Delta} - \cancel{\langle c^\dagger_{i\uparrow}c_{i\downarrow}\rangle\langle c^\dagger_{i\downarrow}c_{i\uparrow}\rangle}$$

  - The mean field Hamiltonian

$$\mathcal{H}_{MF} = -t\sum_{\langle i,j\rangle\sigma} c^\dagger_{i\sigma}c_{j\sigma} - \mu'\sum_{i\sigma} n_{i\sigma} - \sum_i \left(\Delta_i c^\dagger_{i\uparrow}c^\dagger_{i\downarrow} + \boxed{\Delta^*_i} c_{i\downarrow}c_{i\uparrow}\right)$$

On-site pairing

- **Pairing in "unconventional" superconductivity**

$$\mathcal{H}_{MF} = -t\sum_{\langle i,j\rangle\sigma} c^\dagger_{i\sigma}c_{j\sigma} - \mu'\sum_{i\sigma} n_{i\sigma} - \sum_i \left(\Delta_i c^\dagger_{i\uparrow}c^\dagger_{i\downarrow} + \Delta^*_i c_{i\downarrow}c_{i\uparrow}\right) +$$

$$\sum_{\langle i,j\rangle}\left(\Delta_{ij}c^\dagger_{i\uparrow}c^\dagger_{j\downarrow} + \boxed{\Delta^*_{ij}} c_{j\downarrow}c_{i\uparrow}\right)$$

Nearest-neighbor pairing

Role of pairing in Kitaev 1-D Chain

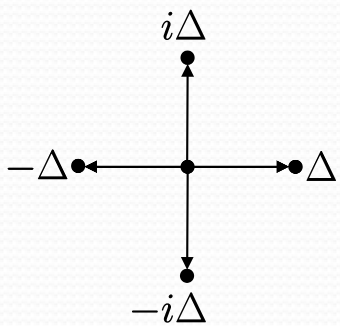
- **Pairing in "unconventional" superconductivity**
  - 2-D lattice model mean field Hamiltonian (lattice constant = 1)

$$\mathcal{H}_{MF} = -t\sum_{\langle i,j\rangle\sigma} c^\dagger_{i\sigma}c_{j\sigma} - \mu'\sum_{i\sigma} n_{i\sigma} + \sum_{\langle i,j\rangle}\left(\Delta_{ij}c^\dagger_{i\uparrow}c^\dagger_{j\downarrow} + \Delta^*_{ij}c_{j\downarrow}c_{i\uparrow}\right)$$

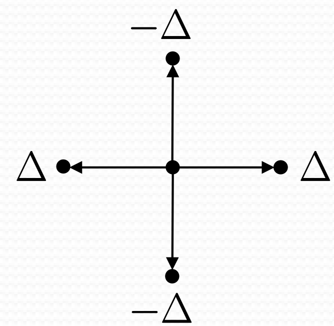
  - Diagonalize

$$c_{i\sigma} = \frac{1}{\sqrt{L_x L_y}}\sum_{\mathbf{k}} e^{ik_x}c_{\mathbf{k}\sigma}$$

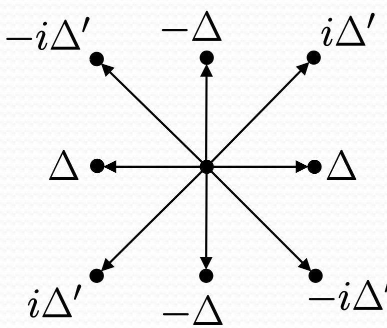
$$c_{j\sigma} = \frac{1}{\sqrt{L_x L_y}}\sum_{\mathbf{k}} e^{ik_y}c_{\mathbf{k}\sigma}$$



p-wave

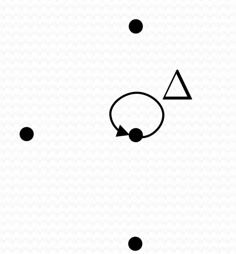


d-wave



d+*id*-wave

$$\Delta(\mathbf{k}) = \begin{cases} 2\Delta\left(\cos(k_x) - \cos(k_y)\right) & d\text{-wave} \\ 2\Delta\left(\cos(k_x) - \cos(k_y) - 4i\Delta'\sin(k_x)\sin(k_y)\right) & d+id\text{-wave} \\ 2i\Delta\left(\sin(k_x) + i\sin(k_y)\right) & p\text{-wave} \end{cases}$$



s-wave

Properties of Majorana Fermions

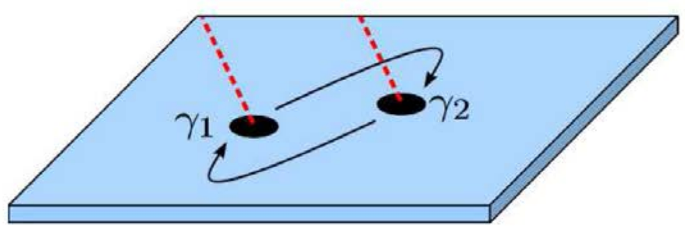
- **Are Majoranas "hard-core balls"?**
  - Majorana "mode" is a superposition of electron and hole states
  - Is this like a bound state? e.g. exciton, hydrogen atom, positronium?
  - Can "count" them by putting them in bins?
  - Sure, define a number operator $n_i^{\mathrm{MF}} = \gamma_i^\dagger \gamma_i = \gamma_i \gamma_i = \gamma_i^2 = 1$
  - Garbage! Okay, counting doesn't make sense!
- **Regular fermion basis**
  - We can count regular fermions $n_i = f_i^\dagger f_i$
  - We can pair Majoranas into regular fermions and measure them
  - How to chose? Number of pairings: $2N!/2!(2N-2)!$
  - Overlap between states $\frac{i}{2} t \gamma_{2i-1}\gamma_{2i} = t\left(n_i - \frac{1}{2}\right)$
  - To observe the state of the system we need to "fuse" two Majoranas

Properties of Majorana Fermions

- **Non-abelian statistics**
  - A system of $2N$ *well separated* Majoranas has a 2^N degenerate ground state. Think of N independent of 1-D Kitaev chains
  - Exchanging or “braiding” connects two different ground states
  - What is nonabelian about them?

“*if one performs sequential exchanges, the final state depends on the order in which they are carried out*”

  - Consider the exchange of two Majoranas

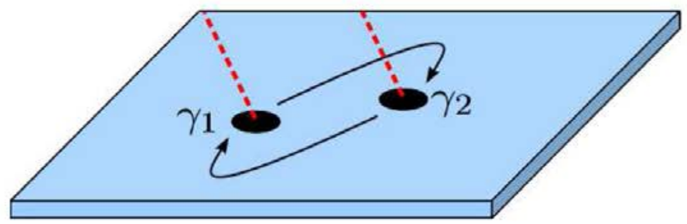


$$\gamma_1 \rightarrow -\gamma_2$$

$$\gamma_2 \rightarrow +\gamma_1$$

Properties of Majorana Fermions

- **Non-abelian statistics**
  - Exchange of two Majoranas



  - Define "braiding" operator $B_{12} = \frac{1}{\sqrt{2}}(1+\gamma_1\gamma_2)$

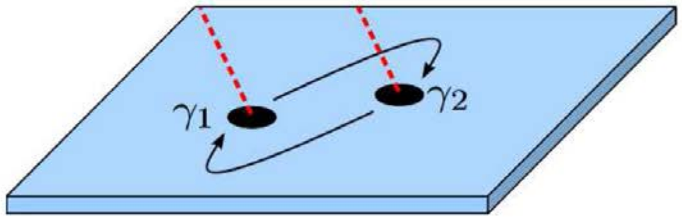
$$\gamma_i \rightarrow B_{12}\gamma_i B_{12}^\dagger$$

$$B_{12}\gamma_i B_{12}^\dagger = \frac{1}{\sqrt{2}}(1+\gamma_1\gamma_2)\,\gamma_i \frac{1}{\sqrt{2}}\left(1+(\gamma_1\gamma_2)^\dagger\right)$$
$$= \frac{1}{2}(1+\gamma_1\gamma_2)\,\gamma_i\left(1+\gamma_2^\dagger\gamma_1^\dagger\right)$$
$$= \frac{1}{2}(1+\gamma_1\gamma_2)\,\gamma_i\,(1+\gamma_2\gamma_1)$$
$$= \frac{1}{2}(\gamma_i+\gamma_1\gamma_2\gamma_i+\gamma_i\gamma_2\gamma_1+\gamma_1\gamma_2\gamma_i\gamma_2\gamma_1)$$

$$\gamma_1 \rightarrow -\gamma_2$$
$$\gamma_2 \rightarrow +\gamma_1$$

Properties of Majorana Fermions

- **Non-abelian statistics**
  - Exchange of two Majoranas



$$B_{12}|0\rangle = \frac{1}{\sqrt{2}}\left[1 + i\left(f_1^\dagger + f_1\right)\left(f_1^\dagger - f_1\right)\right]|0\rangle$$
$$= \frac{1}{\sqrt{2}}\left[1 + if_1^\dagger f_1^\dagger - if_1^\dagger f_1 + if_1 f_1^\dagger - if_1 f_1\right]|0\rangle$$
$$= \frac{1}{\sqrt{2}}\left[|0\rangle + if_1^\dagger f_1^\dagger|0\rangle - 0 + if_1 f_1^\dagger|0\rangle - 0\right]$$
$$= \frac{1}{\sqrt{2}}\left[|0\rangle + 0 + if_1|1\rangle\right]$$
$$= \frac{1}{\sqrt{2}}\left[|0\rangle + i|0\rangle\right]$$
$$= \frac{1}{\sqrt{2}}(1+i)|0\rangle$$

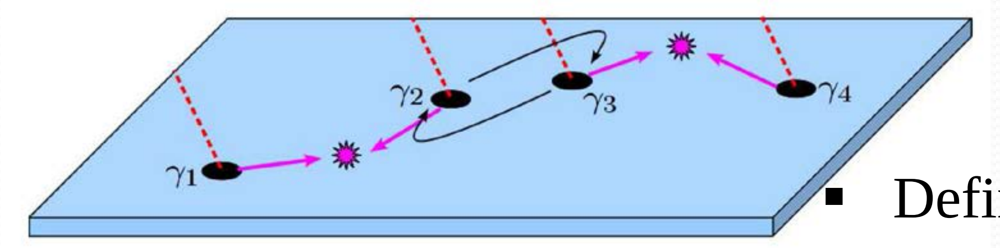
  - Effect on number states

$$B_{12}|0\rangle = \frac{1}{\sqrt{2}}(1+i)|0\rangle$$

$$B_{12}|1\rangle = \frac{1}{\sqrt{2}}(1-i)|1\rangle$$

Properties of Majorana Fermions

- **Non-abelian statistics**
  - Exchange of four Majoranas



  - Effect on number states

$$B_{12}|00\rangle = \frac{1}{\sqrt{2}}(1+i)\,|00\rangle$$

$$B_{34}|00\rangle = \frac{1}{\sqrt{2}}(1+i)\,|00\rangle$$

$$B_{23}|00\rangle = \frac{1}{\sqrt{2}}(|00\rangle + i|11\rangle)$$

- Define Pauli matrices for rotations on the Bloch sphere

$$-i\gamma_1\gamma_2 = \sigma_z, -i\gamma_3\gamma_4 = \sigma_z$$
$$-i\gamma_2\gamma_3 = \sigma_x$$
$$-i\gamma_1\gamma_3 = \sigma_y, -i\gamma_2\gamma_4 = \sigma_y$$

- Braiding as rotations

$$B_{12} = B_{34} = e^{-\frac{i\pi}{4}\sigma_z}$$

$$B_{23} = e^{-\frac{i\pi}{4}\sigma_x}$$

Ingredients for observing Majoranas?

- **Key ingredients**
  - Mechanism for *pairing* of regular fermions
  - *Spin* degree of freedom must be suppressed
  - Additionally we need
    - *p-wave* pairing symmetry
    - *Spin-triplet* state
- **Tools that provide these ingredients**
  - Pairing ➔ in superconductors or proximity effect
  - Suppress spin ➔ break time-reversal symmetry or polarize a band
- **Few important approaches/proposals**
  - Engineer systems with strong spin-orbit coupling and superconductors
  - Induced triplet *p*-wave pairing in non-centrosymmetric superconductors
  - Discover Time Reversal Invariant topological superconductors!

"Artificial" topological superconductors

- **"Spinless" *p*-wave superconductors**
  - 2^{nd} quantized Hamiltonian

$$\mathcal{H}_0 = \sum_{\sigma=\uparrow,\downarrow} \int d^D r \; \Psi_\sigma^\dagger(\mathbf{r}) H_0(\mathbf{r}) \Psi_\sigma(\mathbf{r})$$

$$H_0(\mathbf{r}) = \frac{\mathbf{p}^2}{2m} - \mu + V(\mathbf{r}) + \alpha \left(\mathbf{E}(\mathbf{r}) \times \mathbf{p} \right) \cdot \bar{\sigma} + \frac{1}{2} g \mu_B \mathbf{B}(\mathbf{r}) \cdot \bar{\sigma}$$

  - Pairing Hamiltonian: $\mathcal{H}_S = \int d^D r \; d^D r' \; \Psi_\downarrow(\mathbf{r}) \Delta(\mathbf{r}, \mathbf{r}') \Psi_\uparrow(\mathbf{r}') + \mathrm{h.c.}$
  - Nambu spinor $\bar{\Psi}(\mathbf{r}) = \begin{pmatrix} \Psi_\uparrow(\mathbf{r}) \\ \Psi_\downarrow(\mathbf{r}) \\ \Psi_\downarrow^\dagger(\mathbf{r}) \\ -\Psi_\uparrow^\dagger(\mathbf{r}) \end{pmatrix}$
  - *s*-wave singlet pairing inherited from superconductor: $\Delta(\mathbf{r}, \mathbf{r}')$

“Artificial” topological superconductors

- **“Spinless” *p*-wave superconductors** $\mathcal{H}_0 = \sum_{\sigma=\uparrow,\downarrow} \int d^D r\ \Psi_\sigma^\dagger(\mathbf{r}) H_0(\mathbf{r}) \Psi_\sigma(\mathbf{r})$
  - Recall 1st and 2nd quantized Hamiltonians

$$H_0(\mathbf{r}) = \frac{\mathbf{p}^2}{2m} - \mu + V(\mathbf{r}) + \alpha\left(\mathbf{E}(\mathbf{r}) \times \mathbf{p}\right) \cdot \bar{\sigma} + \frac{1}{2} g \mu_B \mathbf{B}(\mathbf{r}) \cdot \bar{\sigma}$$

  - Define

$$\bar{H}_0(\mathbf{r}) = \begin{pmatrix} H_0(\mathbf{r}) & \hat{0}_\sigma \\ \hat{0}_\sigma & -\sigma_y H_0^*(\mathbf{r}) \sigma_y \end{pmatrix} \qquad \bar{\Delta}(\mathbf{r}, \mathbf{r}') = \begin{pmatrix} \hat{0}_\sigma & \Delta^*(\mathbf{r}, \mathbf{r}') \hat{1}_\sigma \\ \Delta(\mathbf{r}, \mathbf{r}') \hat{1}_\sigma & \hat{0}_\sigma \end{pmatrix}$$

  - Then total Hamiltonian is given by

$$\mathcal{H} = \frac{1}{2} \int d^D r\, d^D r'\ \bar{\Psi}^\dagger(\mathbf{r}) \left[\bar{H}_0(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}') + \bar{\Delta}(\mathbf{r}, \mathbf{r}') \right] \bar{\Psi}(\mathbf{r}') = \frac{1}{2} \sum_i E_i\ \Psi_i^\dagger \Psi_i$$

    where

$$\Psi_i = \int d^D r\ \bar{\psi}_i(\mathbf{r}) \cdot \bar{\Psi}(\mathbf{r}) = \int d^D r\ \bar{\psi}_i^\dagger(\mathbf{r}) \bar{\Psi}(\mathbf{r})$$

"Artificial" topological superconductors

- **Obtaining Bogoliubov-de Gennes (BdG) equation**

$$\int d^Dr\, d^Dr'\; \bar{\Psi}^\dagger(\mathbf{r}) \left[\bar{H}_0(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}') + \bar{\Delta}(\mathbf{r},\mathbf{r}')\right] \bar{\Psi}(\mathbf{r}') = \sum_j E_j\; \Psi_j^\dagger \Psi_j$$

$$= \sum_j E_j \int d^Dr\; \bar{\Psi}^\dagger(\mathbf{r})\bar{\psi}_j(\mathbf{r}) \int d^Dr'\; \bar{\psi}_j^\dagger(\mathbf{r}')\bar{\Psi}(\mathbf{r}')$$

$$= \int d^Dr' \int d^Dr \sum_j E_j\; \bar{\Psi}^\dagger(\mathbf{r})\bar{\psi}_j(\mathbf{r})\bar{\psi}_j^\dagger(\mathbf{r}')\bar{\Psi}(\mathbf{r}')$$

$$= \int d^Dr' \int d^Dr\; \bar{\Psi}^\dagger(\mathbf{r}) \left[\sum_j E_j\; \bar{\psi}_j(\mathbf{r})\bar{\psi}_j^\dagger(\mathbf{r}')\right] \bar{\Psi}(\mathbf{r}')$$

- Compare

$$\bar{H}_0(\mathbf{r}) \int d^Dr'\delta(\mathbf{r}-\mathbf{r}')\bar{\psi}_i(\mathbf{r}') + \int d^Dr'\,\bar{\Delta}(\mathbf{r},\mathbf{r}')\bar{\psi}_i(\mathbf{r}') = \sum_j E_j\; \bar{\psi}_j(\mathbf{r}) \int d^Dr'\bar{\psi}_j^\dagger(\mathbf{r}')\bar{\psi}_i(\mathbf{r}')$$

$$\bar{H}_0(\mathbf{r})\bar{\psi}_i(\mathbf{r}) + \int d^Dr'\,\bar{\Delta}(\mathbf{r},\mathbf{r}')\bar{\psi}_i(\mathbf{r}') = \sum_j E_j\; \bar{\psi}_j(\mathbf{r})\delta_{ij} = E_i\; \bar{\psi}_i(\mathbf{r})$$

"Artificial" topological superconductors

- **Artifacts of the BdG formalism?**
  - Particle-hole symmetry $P = \tau_y \otimes \sigma_y K = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} K$
  - Action on operators

$$P \bar{H}_0(\mathbf{r}) P^\dagger = -\bar{H}_0(\mathbf{r})$$

$$P \bar{\Delta}(\mathbf{r}, \mathbf{r}') P^\dagger = -\bar{\Delta}(\mathbf{r}, \mathbf{r}')$$

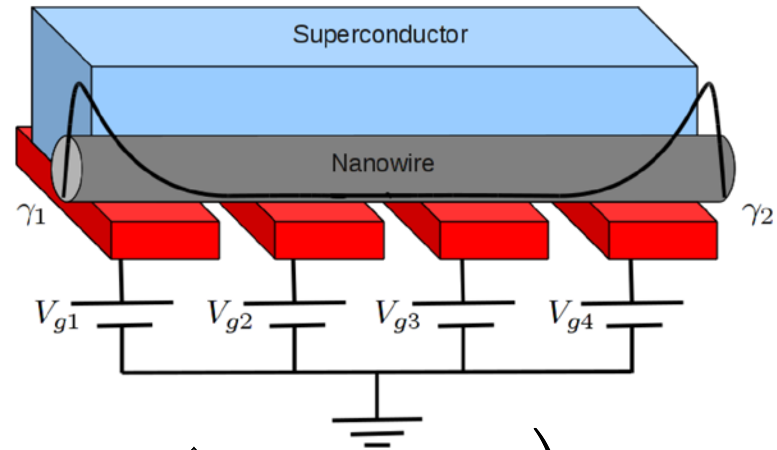
  - Relation between particle and hole eigenstates

$$\bar{\psi}_j(\mathbf{r}) = P \bar{\psi}_i(\mathbf{r}) \qquad \Psi_j = \Psi_i^\dagger$$

  - Therefore, Majorana fermions $\bar{\psi}_M(\mathbf{r}) = P \bar{\psi}_M(\mathbf{r})$
  - Structure of the Nambu spinor

$$\boxed{\bar{\psi}_M(\mathbf{r}) = \begin{pmatrix} f(\mathbf{r}) \\ g(\mathbf{r}) \\ g^*(\mathbf{r}) \\ -f^*(\mathbf{r})) \end{pmatrix}} \quad P \bar{\psi}_M(\mathbf{r}) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} K \begin{pmatrix} f(\mathbf{r}) \\ g(\mathbf{r}) \\ g^*(\mathbf{r}) \\ -f^*(\mathbf{r})) \end{pmatrix} = \begin{pmatrix} f(\mathbf{r}) \\ g(\mathbf{r}) \\ g^*(\mathbf{r}) \\ -f^*(\mathbf{r})) \end{pmatrix}$$

"Artificial" topological superconductors



- **"Spinless" *p*-wave superconductors**
  - 2nd quantized Hamiltonian

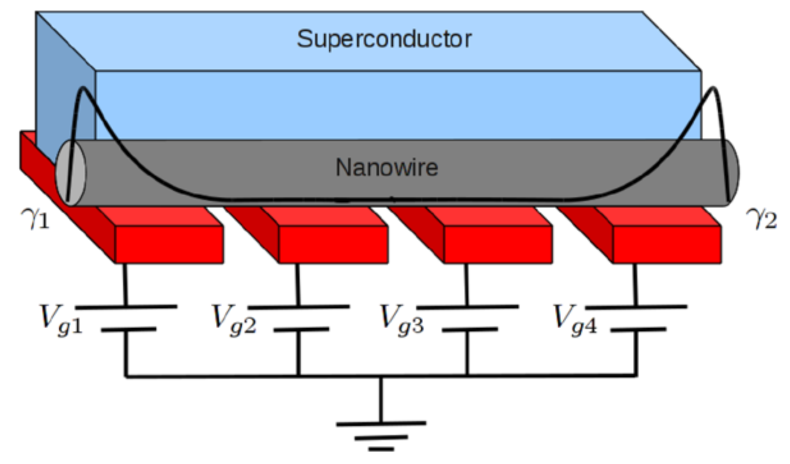
$$H_0(x) = \frac{k_x^2}{2m} - \mu + \tilde{\alpha} k_x \sigma_y + \frac{1}{2}\tilde{B}\sigma_z$$

$$\bar{H}_0(x) = \begin{pmatrix} \frac{k_x^2}{2m} - \mu + \tilde{\alpha} k_x \sigma_y + \frac{1}{2}\tilde{B}\sigma_z & \hat{0}_\sigma \\ \hat{0}_\sigma & -\left(\frac{k_x^2}{2m} - \mu\right) + \tilde{\alpha} k_x \sigma_y + \frac{1}{2}\tilde{B}\sigma_z \end{pmatrix}$$

  - Assume on-site pairing $\bar{\Delta}(x,x') = \bar{\Delta}\delta(x-x')$ where $\bar{\Delta} = \Delta \begin{pmatrix} \hat{0}_\sigma & \hat{1}_\sigma \\ \hat{1}_\sigma & \hat{0}_\sigma \end{pmatrix}$
  - First consider $\Delta = 0$
  - Block diagonal 2 × 2 Hamiltonians

$$H_0(k_x) = \pm\left(\frac{k_x^2}{2m} - \mu\right) + \tilde{\alpha} k_x \sigma_y + \frac{1}{2}\tilde{B}\sigma_z$$

“Artificial” topological superconductors



- **“Spinless” *p*-wave superconductors**
  - Diagonalizing 2 × 2 matrices

$$H_0(k_x) = \pm\left(\frac{k_x^2}{2m} - \mu\right) + \tilde{\alpha}k_x\sigma_y + \frac{1}{2}\tilde{B}\sigma_z$$

$$\left[H_0(k_x) \mp \left(\frac{k_x^2}{2m} - \mu\right)\right]^2 = \left(\tilde{\alpha}k_x\sigma_y + \frac{1}{2}\tilde{B}\sigma_z\right)^2$$

$$= (\tilde{\alpha}k_x)^2\sigma_y^2 + \frac{1}{4}\tilde{B}^2\sigma_z^2 + \frac{1}{2}\tilde{\alpha}k_x\tilde{B}\{\sigma_y, \sigma_z\}$$

$$\boxed{\{\sigma_i, \sigma_j\} = 2\delta_{ij}}$$

$$= (\tilde{\alpha}k_x)^2 + \frac{1}{4}\tilde{B}^2$$

- Eigenvalues

$$E_{\pm\pm}(k_x) = \pm\left(\frac{k_x^2}{2m} - \mu\right) \pm\sqrt{(\tilde{\alpha}k_x)^2 + \frac{1}{4}\tilde{B}^2}$$

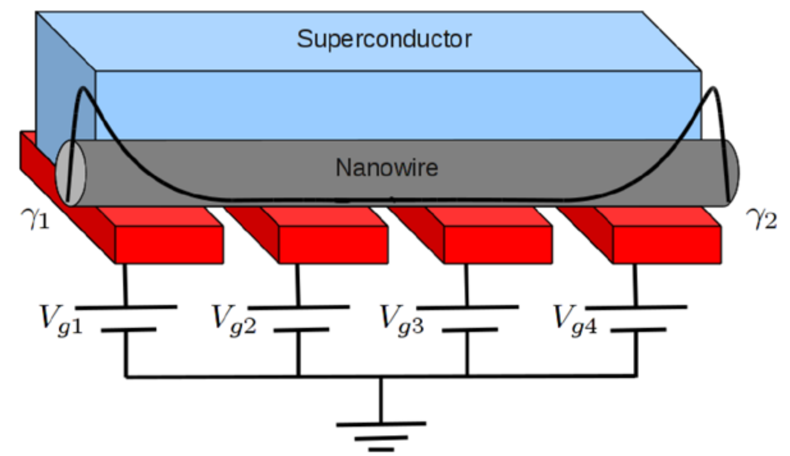
“Artificial” topological superconductors

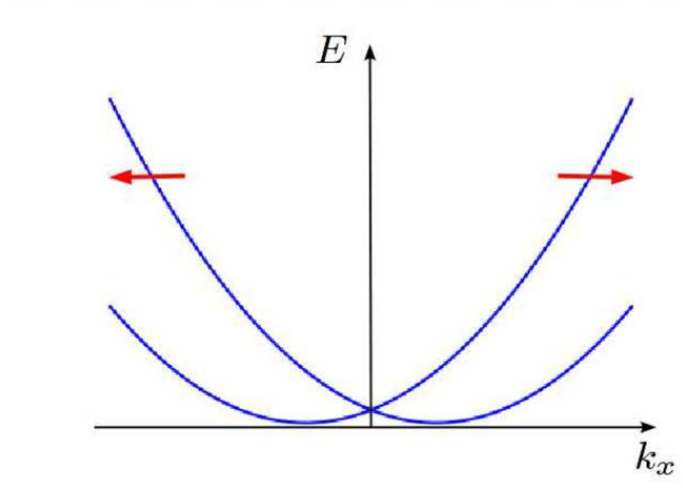
- **“Spinless” *p*-wave superconductors**
  - Recall eigenvalues

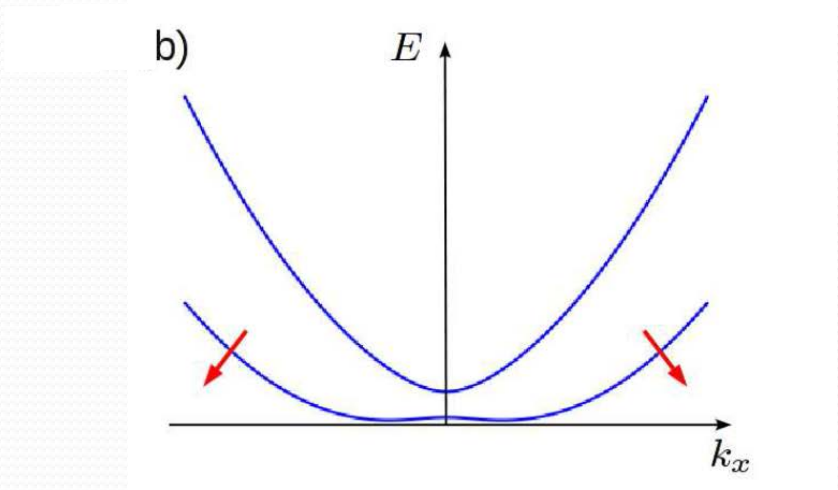
$$E_{\pm\pm}(k_x) = \pm\left(\frac{k_x^2}{2m} - \mu\right) \pm\sqrt{(\tilde{\alpha}k_x)^2 + \frac{1}{4}\tilde{B}^2}$$

  - For no Zeeman field

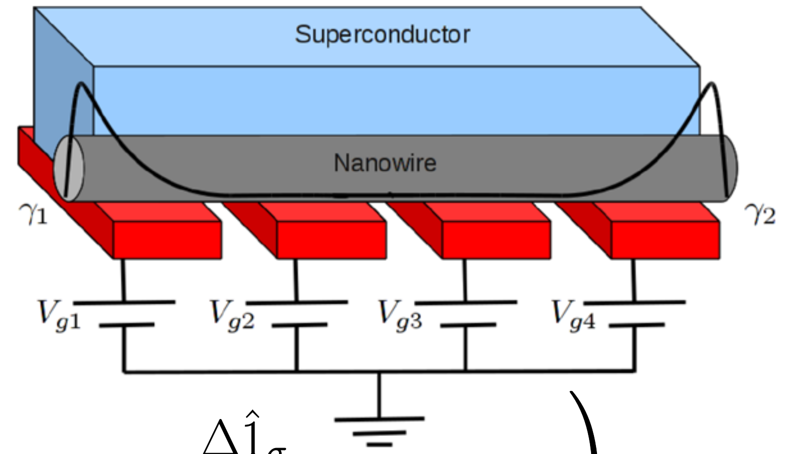
$$E_{+\pm}(k_x) = \left(\frac{k_x^2}{2m} - \mu\right) \pm\tilde{\alpha}k_x$$







“Artificial” topological superconductors



- **“Spinless” *p*-wave superconductors**
  - Now, slowly turn on the pairing $\Delta > 0$
  - Total Hamiltonian becomes

$$\bar{H}_0(x)+\bar{\Delta} = \begin{pmatrix} \frac{k_x^2}{2m} - \mu + \tilde{\alpha} k_x \sigma_y + \frac{1}{2}\tilde{B}\sigma_z & \Delta \hat{1}_\sigma \\ \Delta \hat{1}_\sigma & -\left(\frac{k_x^2}{2m} - \mu\right) + \tilde{\alpha} k_x \sigma_y + \frac{1}{2}\tilde{B}\sigma_z \end{pmatrix}$$

  - Brute force diagonalization

$$E_{\pm\pm} = \pm \left[\left(\left(\frac{k_x^2}{2m} - \mu \right)^2 + \Delta^2 \right)^{1/2} \pm \left(\tilde{\alpha}^2 k_x^2 + \frac{\tilde{B}^2}{4} \right)^{1/2} \right]$$

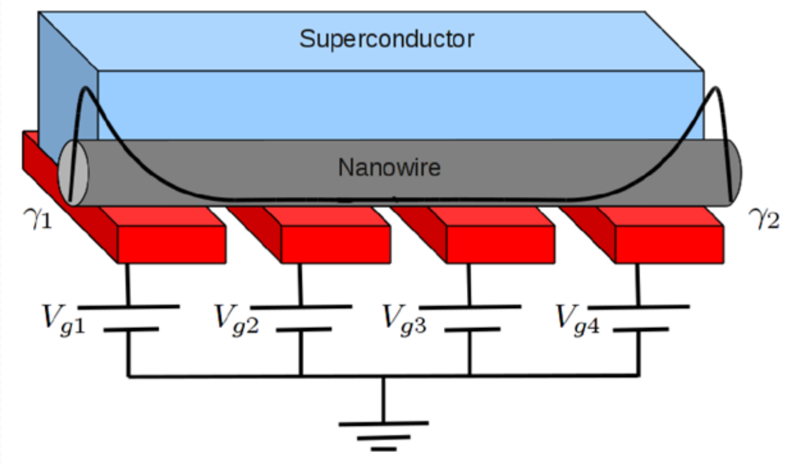
  - The gap vanishes at

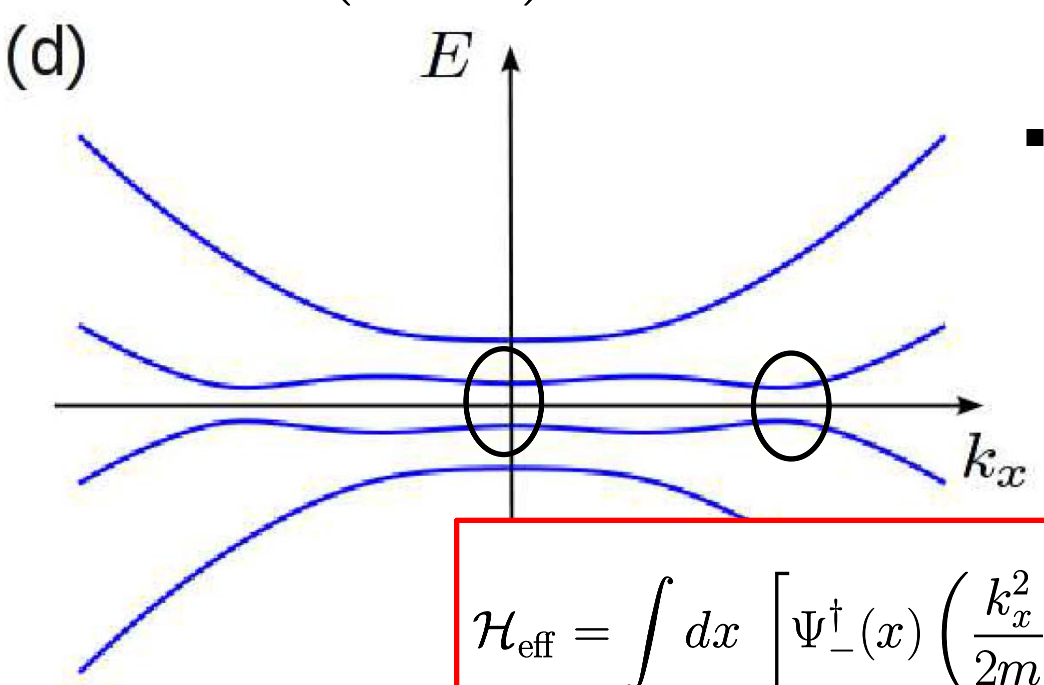
$$\left(\frac{k_x^2}{2m} - \mu \right)^2 + \Delta^2 = \tilde{\alpha}^2 k_x^2 + \frac{\tilde{B}^2}{4} \qquad \tilde{B}^2 = 4\left(\mu^2 + \Delta^2\right)$$

“Artificial” topological superconductors

- **“Spinless” *p*-wave superconductors**
  - The gap vanishes at $\tilde{B}^2 = 4\left(\mu^2 + \Delta^2\right)$

    and $$\left(\frac{k_x^2}{2m} - \mu\right)^2 + \Delta^2 = \tilde{\alpha}^2 k_x^2 + \frac{\tilde{B}^2}{4}$$





- Consider the limit

  $$\tilde{B}| \gg E_{so}, |\Delta|, \quad \mu = 0$$

  where $E_{so} = m\tilde{\alpha}^2/2$

  $$\Psi_-(x) \approx \Psi_\downarrow(x)$$

$$\boxed{\mathcal{H}_{\text{eff}} = \int dx \left[\Psi_-^\dagger(x)\left(\frac{k_x^2}{2m} - \frac{1}{2}\tilde{B}\right)\Psi_-(x) + i\frac{\tilde{\alpha}}{\tilde{B}}\Delta\Psi_-(x)k_x\Psi_-(x)\right]}$$

Conclusions and Outlook

- **Overview**
  - How to obtain Majorana fermions
  - Non-abelian statistics
  - Engineering/finding systems that host Majorana zero modes
- **Experimental progress**
  - Kouwenhoven group first to see “zero bias conductance peak” (ZBCP) in InSb nanowires
  - Other groups confirmed existence of ZBCP with different experimental parameters
- **Experimental to-do’s**
  - Verify non-abelian statistics
  - Test more platforms for hosting Majorana fermions
  - Accomplish reliable quantum computation

References

- Martin Leijnse and Karsten Flensberg, “Introduction to topological superconductivity and Majorana fermions,” *Semiconductor Science and Technology*, vol. 27, no. 12, p. 124003, 2012

- Jason Alicea, “New directions in the pursuit of Majorana fermions in solid state systems,” *Reports on Progress in Physics*, vol. 75, no. 7, p. 076501, 2012

Thanks for listening!

Questions?