The Quantum Spin Hall Effect

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Outline

Classification of Phases of Matter

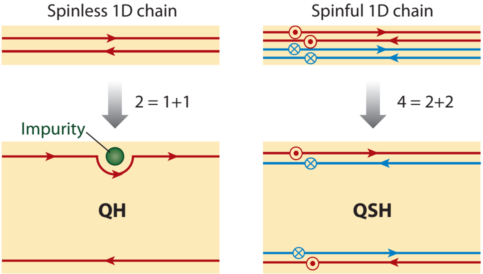
- **Ginzburg-Landau Theory of Phase Transitions**
  - Classify phases based on which symmetries they break
  - More rigorous definition of "symmetry breaking": ground state does not possess symmetries of the Hamiltonian
  - Ordered phase characterized by **local** order parameter
- **Phases Defined by Symmetry Breaking**
  - Rotational and Translational: Crystalline Solids (*continuous to discrete*)
  - Spin Rotation Symmetry: Ferromagnets and Antiferromagnets
- **Counter examples**
  - Integer and Fractional Quantum Hall Effects
  - Certain Spin Liquids
- **Topological Phases of Quantum Matter**
  - Topological phases characterized by an invariant quantity: TKNN number or Chern number
  - Chern number is equal to the number of **stable** gapless edge states

"Topological Protection" of Edge States

- **What is "Topological" about this new Quantum Phase?**
  - The bulk topology is responsible for *fractionalization* on the edge
  - Degrees of freedom of the electron states are **not** localized
  - Failure to define local order parameter makes sense
- **Example: The Integer Quantum Hall (QH) Effect**
  - Area of closed orbits in the bulk becomes quantized, bulk electrons become localized, and the bulk turns into an insulator
  - The skipping edge orbits form extended one-dimensional channels with a quantized conductance of e^2/h per channel
  - Different values of the Hall conductance σ_{xy} are distinct phases of matter
  - Different σ_{xy} cannot be adiabatically connected to each other without closing a spectral gap

The Quantum Spin Hall Effect (QSHE)

- Conceptual analogy between the quantum Hall and quantum spin Hall effects

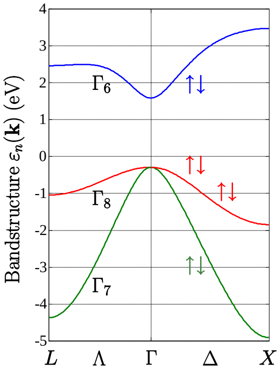


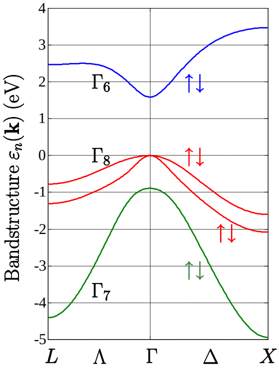
- Transverse spin conductance

$$\sigma_{xy}^{\text{spin}} = \nu\left(\frac{2e^2}{h}\right) \qquad \nu = 0, 1$$

Bandstructure of CdTe

- s-like (conduction) band Γ_6 and p-like (valence) bands Γ_7 and Γ_8 with (right) and without (left) turning on spin-orbit interaction

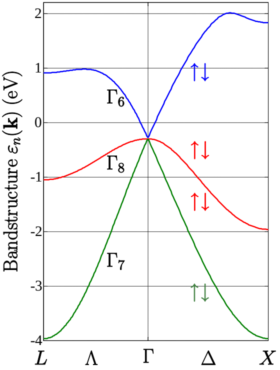


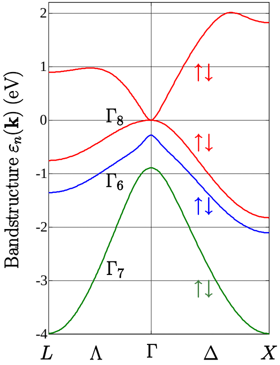


- With spin-orbit interaction Γ_8 splits into the Light Hole (LH) and Heavy Hole (HH) bands away from the Γ point
- The split-off band Γ_7 shifts downward

Bandstructure of HgTe

- s-like (conduction) band Γ_6 and p-like (valence) bands Γ_7 and Γ_8 with (right) and without (left) turning on spin-orbit interaction

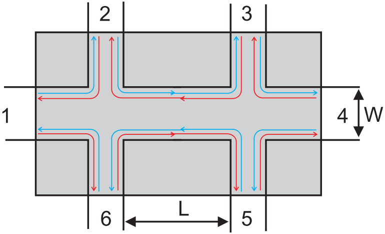


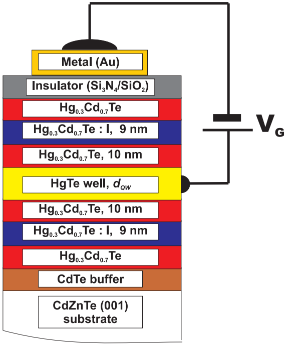


- The Γ_8 splits into LH and HH like CdTe except the LH band is inverted
- The ordering of LH band in Γ_8 and Γ_6 bands are switched

Quantum Well Fabrication

- Molecular Beam Epitaxy (MBE) grown HgTe/CdTe quantum well structure
- Confinement in (say) the z-direction

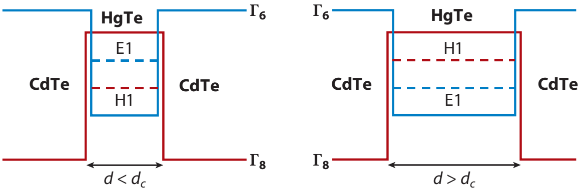




- $L = 600\ \mu m$ and $W = 200\ \mu m$
- Gate voltage V_G used to tune the Fermi level in HgTe quantum well

Topological Phase Transition

- For $d_{QW} > d_c$ the HgTe layer becomes quantum spin Hall insulator



- The E_1 and H_1 subbands switch to *inverted* ordering for $d_{QW} > d_c$ just like in bulk HgTe
- Quantum confinement does **not** help create a topologically nontrivial phase
- Why not just use bulk HgTe then?
- There is **no gap** in (unstrained) bulk HgTe!

The Bernevig-Hughes-Zhang Model

- **Construction of a simple 2D lattice model**
  - Define a new basis from two spaces: **orbital** (τ_i) and **spin** (σ_i) with $i = 0, 1, 2, 3$

$$|\psi\rangle = \begin{pmatrix} \left|E_1, +\frac{1}{2}\right\rangle & \left|E_1, -\frac{1}{2}\right\rangle & \left|H_1, +\frac{1}{2}\right\rangle & \left|H_1, -\frac{1}{2}\right\rangle \end{pmatrix}^T$$

  - Must respect symmetries of the system: **time-reversal** and **inversion**

$$\hat{\Theta} = i\tau_0 \otimes \sigma_y \mathcal{K} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \mathcal{K}, \; \hat{\mathcal{P}} = \tau_0 \otimes \sigma_y = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- **Representation of *arbitrary* 4×4 Hamiltonian matrix**

$$\begin{aligned} \mathcal{H}(\mathbf{k}) &= \varepsilon(\mathbf{k})\mathbb{I}_{4\times 4} + \sum_{i=0}^{4} d_i(\mathbf{k})\Gamma_i + \sum_{i<j=0}^{4} d_{ij}(\mathbf{k})\Gamma_{ij} \qquad i, j = 0, 1, 2, 3, 4 \\ \{\Gamma_i, \Gamma_j\} &= 2\delta_{ij}\mathbb{I}_{4\times 4} \\ \Gamma_{ij} &= \frac{1}{2i}\left[\Gamma_i, \Gamma_j\right] \end{aligned}$$

The Bernevig-Hughes-Zhang Model

- Definition of Γ_i's ($4 \times 4 - 1 = 15$ choices!)

$$\{\Gamma_0, \Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4\} \equiv \{\tau_z \otimes \sigma_0, -\tau_x \otimes \sigma_x, -\tau_x \otimes \sigma_y, -\tau_x \otimes \sigma_z, \tau_y \otimes \sigma_0\}$$

- Check symmetries of the $\{\Gamma_i\}$

	Γ_0	Γ_1	Γ_2	Γ_3	Γ_4
$\hat{\Theta}$	+	−	−	−	−
$\hat{\mathcal{P}}$	+	−	−	−	−
$\hat{\Theta}\hat{\mathcal{P}}$	+	+	+	+	+

- Rotation of π about x, y, and z-axis

	Γ_0	Γ_1	Γ_2	Γ_3	Γ_4
$\hat{\mathcal{R}}_x(\pi)$	+	+	−	−	+
$\hat{\mathcal{R}}_y(\pi)$	+	−	+	−	+
$\hat{\mathcal{R}}_z(\pi)$	+	−	−	+	+

Model Hamiltonian

- The Hamiltonian on a simple-cubic lattice

$$\mathcal{H}(\mathbf{k}) = \varepsilon(\mathbf{k})\mathbb{I}_{4\times 4} + M\Gamma_0 + \sum_{i=1}^{3}\left(\alpha\cos\left(k_i a\right)\Gamma_0 + \beta\sin\left(k_i a\right)\Gamma_i\right)$$

- In 2-D HgTe quantum wells (fix k_z)

$$\begin{aligned}\mathcal{H}(\mathbf{k}) &= \varepsilon(\mathbf{k})\mathbb{I}_{4\times 4} + (M-2B)\Gamma_0 - 2B\cos\left(k_x a\right)\Gamma_0 \\ &\quad -2B\cos\left(k_y a\right)\Gamma_0 + A\sin\left(k_x a\right)\Gamma_1 + A\sin\left(k_y a\right)\Gamma_2\end{aligned}$$

- Ignore $\varepsilon(\mathbf{k})$, setting $a = 1$, and defining $\mathcal{M}(\mathbf{k}) = M - 2B\left(2 - \cos\left(k_x\right) - \cos\left(k_y\right)\right)$

$$\mathcal{H}(\mathbf{k}) = \mathcal{M}(\mathbf{k})\Gamma_0 + A\sin\left(k_x\right)\Gamma_1 + A\sin\left(k_y\right)\Gamma_2$$

- Full Hamiltonian (already diagonal in $\mathbf{k}$)

$$\mathcal{H} = \sum_{\mathbf{k}}\left(\mathcal{M}(\mathbf{k})\Gamma_0 + A\sin\left(k_x\right)\Gamma_1 + A\sin\left(k_y\right)\Gamma_2\right)$$

Solution of Bulk States

- Neat trick to diagonalize

$$\begin{aligned}\mathcal{H}^2(\mathbf{k}) = \mathcal{M}^2(\mathbf{k})\Gamma_0^2 + A^2\sin^2(k_x)\,\Gamma_1^2 + A^2\sin^2(k_y)\,\Gamma_2^2 \\ + A\sin(k_x)\,\mathcal{M}(\mathbf{k})\,\{\Gamma_0,\Gamma_1\} \\ + A\sin(k_y)\,\mathcal{M}(\mathbf{k})\,\{\Gamma_0,\Gamma_2\} \\ + A^2\sin(k_x)\sin(k_y)\,\{\Gamma_1,\Gamma_2\}\end{aligned} \tag{1}$$

- Recall commutation relations

$$\{\Gamma_i,\Gamma_j\} = 2\delta_{ij}\mathbb{I}_{4\times 4} \tag{2}$$

- Using (2) in (1) we get

$$\begin{aligned}\mathcal{H}^2(\mathbf{k}) &= \left(\mathcal{M}^2(\mathbf{k}) + A^2\sin^2(k_x) + A^2\sin^2(k_y)\right)\mathbb{I}_{4\times 4} \\ E_\pm(\mathbf{k}) &= \pm\sqrt{\mathcal{M}^2(\mathbf{k}) + A^2\sin^2(k_x) + A^2\sin^2(k_y)}\end{aligned}$$

Explicit Solution of Edge States

- Finite thickness in the y-direction

$$c_{\mathbf{k}} \equiv c_{k_x,k_y} = \frac{1}{L}\sum_j e^{ik_y j} c_{k_x,j}$$

$$\mathcal{H} = \sum_{\mathbf{k}} \left(A\sin(k_x)\Gamma^1 + A\sin(k_y)\Gamma^2 + \left(M - 2B\left(2 - \cos(k_x) - \cos(k_y)\right)\right)\Gamma^5\right) c^\dagger_{\mathbf{k}} c_{\mathbf{k}}$$

$$\begin{aligned}
\mathcal{H} = &\frac{1}{L}\sum_{k_x}\sum_j \left[A\sin(k_x)\Gamma^1 + \left(M - 4B + 2B\cos(k_x)\right)\Gamma^5\right] c^\dagger_{k_x,j} c_{k_x,j} \\
&+ \frac{1}{L}\sum_{k_x}\sum_j \left(-\frac{iA}{2}\Gamma^2 + B\Gamma^5\right) c^\dagger_{k_x,j+1} c_{k_x,j} \\
&+ \frac{1}{L}\sum_{k_x}\sum_j \left(\frac{iA}{2}\Gamma^2 + B\Gamma^5\right) c^\dagger_{k_x,j-1} c_{k_x,j}
\end{aligned}$$

Explicit Solution of Edge States

- Dropping the subscript we can simply set $k \equiv k_x$

$$\mathcal{H} = \frac{1}{L}\sum_{k,j}\left(\mathcal{M}(k)c^\dagger_{k,j}c_{k,j} + \mathcal{T}c^\dagger_{k,j}c_{k,j+1} + \mathcal{T}^\dagger c^\dagger_{k,j+1}c_{k,j}\right)$$
$$\equiv \sum_{k,k'}\mathcal{H}(k)\delta_{k,k'}$$

- The Schrodinger equation can be written as

$$\mathcal{H}|\psi\rangle = E|\psi\rangle$$

- Since $\mathcal{H}(k)\delta_{k,k'}$ is already diagonal, we get a set of *decoupled* Schrodinger equations in k

$$\mathcal{H}(k)|\psi(k)\rangle = E(k)|\psi(k)\rangle$$

$$|\psi\rangle = \bigotimes_k |\psi(k)\rangle, \qquad E = \sum_k E(k)$$

$$|\psi(k)\rangle = \sum_j \psi(k,j)c^\dagger_j|0\rangle$$

Explicit Solution of Edge States

- Schrodinger equation

$$\mathcal{H}(k)|\psi(k)\rangle = E(k)|\psi(k)\rangle, \qquad |\psi(k)\rangle = \sum_j \psi(k,j) c_j^\dagger |0\rangle$$

- Use the following ansatz ($\lambda \in \mathbb{C}$)

$$\psi(k,j) = \lambda^{-j}\phi(k)$$

- Plugging in the ansatz into the Schrodinger equation

$$\begin{aligned}\langle\psi(k)|\mathcal{H}(k)|\psi(k)\rangle &= \sum_{j,\ell',\ell} \phi^\dagger(k)\mathcal{M}(k)\phi(k)\lambda^{\ell'-\ell}\langle 0|c_{\ell'}c_{k,j}^\dagger c_{k,j} c_\ell^\dagger|0\rangle \\ &+ \sum_{j,\ell',\ell} \phi^\dagger(k)\mathcal{T}\phi(k)\lambda^{\ell'-\ell}\langle 0|c_{\ell'}c_{k,j}^\dagger c_{k,j+1} c_\ell^\dagger|0\rangle \\ &+ \sum_{j,\ell',\ell} \phi^\dagger(k)\mathcal{T}^\dagger\phi(k)\lambda^{\ell'-\ell}\langle 0|c_{\ell'}c_{k,j+1}^\dagger c_{k,j} c_\ell^\dagger|0\rangle\end{aligned}$$

Explicit Solution of Edge States

- The different matrix elements can be calculated

$$\begin{aligned}\langle 0|c_{\ell'} c^\dagger_{k,j} c_{k,j} c^\dagger_\ell|0\rangle &= \delta_{\ell',j}\delta_{\ell,j} \\ \langle 0|c_{\ell'} c^\dagger_{k,j} c_{k,j+1} c^\dagger_\ell|0\rangle &= \delta_{\ell',j}\delta_{\ell,j+1} \\ \langle 0|c_{\ell'} c^\dagger_{k,j+1} c_{k,j} c^\dagger_\ell|0\rangle &= \delta_{\ell',j+1}\delta_{\ell,j}\end{aligned}$$

- Plugging this back we get

$$\langle\psi(k)|\mathcal{H}(k)|\psi(k)\rangle = \sum_{j,\ell',\ell} \phi^\dagger \mathcal{M}(k)\phi\lambda^{\ell'-\ell}\delta_{\ell',j}\delta_{\ell,j} + \sum_{j,\ell',\ell} \phi^\dagger \mathcal{T}\phi\lambda^{\ell'-\ell}\delta_{\ell',j}\delta_{\ell,j+1} + \sum_{j,\ell',\ell} \phi^\dagger \mathcal{T}^\dagger\phi\lambda^{\ell'-\ell}\delta_{\ell',j+1}\delta_{\ell,j}$$

- The sum over j, ℓ' and ℓ goes away to give

$$\langle\psi(k)|\mathcal{H}(k)|\psi(k)\rangle = \phi^\dagger(k)\mathcal{M}(k)\phi(k) + \lambda^{-1}\phi^\dagger(k)\mathcal{T}\phi(k) + \lambda\phi^\dagger(k)\mathcal{T}^\dagger\phi(k) = E(k)$$

Explicit Solution of Edge States

- Simplified Schrodinger equation

$$\lambda^{-1}\mathcal{T}\phi(k) + \lambda\mathcal{T}^{\dagger}\phi(k) + \mathcal{M}(k)\phi(k) = E(k)\phi(k)$$

- Plugging in the explicit expressions for $\mathcal{M}(k)$ and $\mathcal{T}$ we get

$$\lambda^{-1}\left(\frac{iA}{2}\Gamma^2 + B\Gamma^5\right)\phi(k) + \lambda\left(-\frac{iA}{2}\left(\Gamma^2\right)^{\dagger} + B\left(\Gamma^5\right)^{\dagger}\right)\phi(k)$$
$$+\left(A\sin(k)\Gamma^1 - 2B\left[2 - \frac{M}{2B} - \cos(k)\right]\Gamma^5\right)\phi(k) = E(k)\phi(k)$$

- Multiplying both sides by Γ^5 on the right side we get

$$\left(\left(\lambda^{-1} - \lambda\right)\frac{iA}{2}\Gamma^5\Gamma^2 + \left(\lambda^{-1} + \lambda\right)B\right)\phi(k) - 2B\left[2 - \frac{M}{2B} - \cos(k)\right]\phi(k)$$
$$= \left[E(k) - A\sin(k)\Gamma^1\right]\Gamma^5\phi(k)$$

Explicit Solution of Edge States

- Two eigenvalue equations

$$i\Gamma^5\Gamma^2\phi(k) = \underbrace{\left[\frac{2}{A\left(\lambda^{-1}-\lambda\right)}\left\{-\left(\lambda^{-1}+\lambda\right)B + 2B\left[2-\frac{M}{2B}-\cos(k)\right]\right\}\right]}_{\pm 1}\phi(k)$$

$$\Gamma^1\phi(k) = \underbrace{\left[\frac{E(k)}{A\sin(k)}\right]}_{\pm 1}\phi(k)$$

- Solve quadratic equation

$$\lambda_{\pm,\{1,2\}} = \frac{1}{(2B \mp A)}\left\{[4B - M - 2B\cos(k)] + \{+,-\}\sqrt{[4B-M-2B\cos(k)]^2 + A^2 - 4B^2}\right\}$$

- Edge dispersion

$$E(k) = \pm A\sin(k)$$

Explicit Solution of Edge States

- Define new quantity $m(k, M)$ in the expression for λ

$$\lambda_{\pm,\{1,2\}} = \frac{1}{(2B \mp A)} \left\{ \underbrace{[4B - M - 2B\cos(k)]}_{m(k,M)} + \{+,-\}\sqrt{[4B - M - 2B\cos(k)]^2 + A^2 - 4B^2} \right\}$$

- Recall ansatz

$$\psi(k, j) = \lambda^{-j}\phi(k)$$

- Normalization condition on the bottom edge

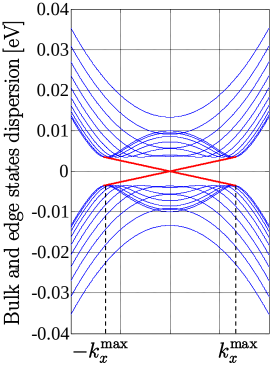
$$|\lambda| > 1$$

- Edge states can only exist for

$$-2B < m(k, M) < 2B$$

Explicit Solution of Edge States

- Bulk dispersion (blue)
  - $E_{\pm}(\mathbf{k}) = \pm\sqrt{A^2\left(\sin^2(k_x) + \sin^2(k_y)\right) + M^2(\mathbf{k})}$
  - $\mathcal{M}(\mathbf{k}) = M - 2B\left(2 - \cos(k_x) - \cos(k_y)\right)$
- Surface dispersion (red)
  - $E_s(k_x) = -As\sin(k_x)$
  - $s = \pm 1$ $(\uparrow\downarrow)$
- $E_s(k_x)$ is valid for $-k_x^{\max} < k < k_x^{\max}$ and the bulk bands have *negative* effective mass, where $k_x^{\max} = \cos^{-1}(1 - M/2B)$

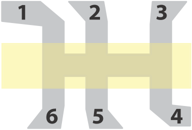


- Given parameters for 7 nm HgTe: $A = 3.65$, $B = -68.6$, $M = -0.010$
- Actual parameters: $A = 0.265$, $B = -55.6$, $M = -0.010$

The Landauer-Büttiker formalism

- The current I_i in a contact i (at the chemical potential μ_i) using the Landauer-Büttiker formula

$$I_i = \frac{e}{h}\sum_j (T_{ji}\mu_i - T_{ij}\mu_j)$$

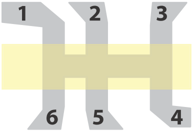


- T_{ij} is the trasmission probability for the electron to go from contacts $i \to j$
- There is perfect transmission between consecutive contacts (other $T_{ij} = 0$)

$$\begin{aligned} T_{i,i+1} &= T_{i+1,i} \\ &= 1 \end{aligned}$$

- For current passed from contacts $k \to \ell$, i.e. $I_k = -I_\ell \equiv I_{k\ell}$ (rest of the $I_i = 0$) we get a set of *linear* coupled equations in μ_i

The Landauer-Büttiker formalism



- Say current is passed from contacts $1 \to 4$, i.e. $I_1 = -I_4 \equiv I_{14}$
- We get six equations in six unknowns μ_i

$$
\begin{aligned}
I_1 &= \frac{e}{h}(\mu_6 + \mu_2 - 2\mu_1) = I_{14} \\
I_2 &= \frac{e}{h}(\mu_1 + \mu_3 - 2\mu_2) = 0 \\
I_3 &= \frac{e}{h}(\mu_2 + \mu_4 - 2\mu_3) = 0 \\
I_4 &= \frac{e}{h}(\mu_3 + \mu_5 - 2\mu_4) = -I_{14} \\
I_5 &= \frac{e}{h}(\mu_4 + \mu_6 - 2\mu_5) = 0 \\
I_6 &= \frac{e}{h}(\mu_5 + \mu_1 - 2\mu_6) = 0
\end{aligned}
$$

The Landauer-Büttiker formalism

- The set of coupled equations can be compactly written as

$$\frac{e}{h}\underbrace{\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \\ 1 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \\ \mu_6 \end{pmatrix}}_{x} = I_{14}\underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}}_{b}$$

- But $\det(A) = 0$! Need to fix one μ_i as reference
- The insolubility ($\det(A) = 0$) is due to redundancy, i.e. not all μ_i's can be treated as unknowns

The Landauer-Büttiker formalism

- Say we set $\mu_4 = 0$ (or gound)

$$\frac{e}{h}\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \\ 1 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ 0 \\ \mu_5 \\ \mu_6 \end{pmatrix} = I_{14}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

- Remove extra equation by performing row reduction

$$\frac{e}{h}\begin{pmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & -2 \end{pmatrix}\begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_5 \\ \mu_6 \end{pmatrix} = I_{14}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

The Landauer-Büttiker formalism

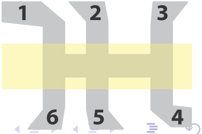
- Solution of Matrix Equation

$$\begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_5 \\ \mu_6 \end{pmatrix} = \frac{I_{14}h}{e} \begin{pmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & -2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_5 \\ \mu_6 \end{pmatrix} = \frac{I_{14}h}{e} \begin{pmatrix} -3/2 \\ -1 \\ -1/2 \\ -1/2 \\ -1 \end{pmatrix}$$

- Voltage difference across contacts i and j

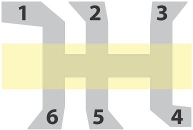
$$V_{ij} = \frac{1}{(-e)} \left(\mu_i - \mu_j\right)$$



The Landauer-Büttiker formalism

- Computation of two-terminal resistance $R_{14,14}$
  - The voltage V_{14} is given by

$$\begin{aligned} V_{14} &= \frac{1}{(-e)}(\mu_1 - \mu_4) \\ &= \frac{1}{(-e)}\left(-\frac{3I_{14}h}{2e} - 0\right) \\ &= \left(\frac{3h}{2e^2}\right) I_{14} \end{aligned}$$



- Therefore, the two-terminal resistance is

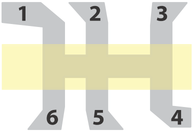
$$\begin{aligned} R_{14,14} &\equiv \frac{V_{14}}{I_{14}} \\ &= \frac{3h}{2e^2} \end{aligned}$$

The Landauer-Büttiker formalism

- Computation of four-terminal resistance $R_{14,23}$

  - The voltage V_{23} is given by

$$\begin{aligned} V_{23} &= \frac{1}{(-e)}(\mu_2 - \mu_3) \\ &= \frac{1}{(-e)}\left(-\frac{I_{14}h}{e} - \left(-\frac{I_{14}h}{2e}\right)\right) \\ &= \left(\frac{h}{2e^2}\right) I_{14} \end{aligned}$$



- The four-terminal resistance is

$$\begin{aligned} R_{14,23} &\equiv \frac{V_{23}}{I_{14}} \\ &= \frac{h}{2e^2} \end{aligned}$$

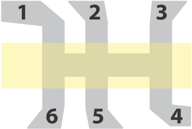
The Landauer-Büttiker formalism

- **Changing contacts**
  - For current passed between contacts 1 and 3, i.e. $I_1 = -I_3 = I_{13}$ we get a matrix equation (rest of the $I_i = 0$)

$$\underbrace{\frac{e}{h}\begin{pmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & -2 \end{pmatrix}}_{A}\underbrace{\begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_5 \\ \mu_6 \end{pmatrix}}_{x} = I_{13}\underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}}_{b}$$

- **Different Convention**

- We only changed b compared to the previous matrix equation
- Ideally we should set $\mu_3 = 0$ instead of μ_4. But this won't matter much



The Landauer-Büttiker formalism

- Solution of Matrix Equation

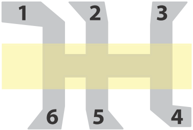
$$\begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_5 \\ \mu_6 \end{pmatrix} = \frac{I_{13}h}{e} \begin{pmatrix} -2 & 1 & 0 & 0 & 1 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & -2 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_5 \\ \mu_6 \end{pmatrix} = \frac{I_{13}h}{e} \begin{pmatrix} -1 \\ -1/3 \\ 1/3 \\ -1/3 \\ -2/3 \end{pmatrix}$$

The Landauer-Büttiker formalism

- Computation of two-terminal resistance $R_{13,13}$

  - The voltage V_{13} is given by



$$
\begin{aligned}
V_{13} &= \frac{1}{(-e)}(\mu_1 - \mu_3) \\
&= \frac{1}{(-e)}\left(-\frac{I_{13}h}{e} - \frac{I_{13}h}{3e}\right) \\
&= \left(\frac{4h}{3e^2}\right) I_{13}
\end{aligned}
$$

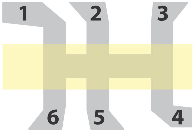
- Therefore, the two-terminal resistance is

$$
\begin{aligned}
R_{13,13} &\equiv \frac{V_{13}}{I_{13}} \\
&= \frac{4h}{3e^2}
\end{aligned}
$$

The Landauer-Büttiker formalism

- Computation of four-terminal resistance $R_{13,56}$
  - The voltage V_{56} is given by

$$\begin{aligned} V_{56} &= \frac{1}{(-e)}(\mu_5 - \mu_6) \\ &= \frac{1}{(-e)}\left(-\frac{I_{13}h}{3e} - \left(-\frac{2I_{13}h}{3e}\right)\right) \\ &= \left(\frac{h}{3e^2}\right) I_{13} \end{aligned}$$



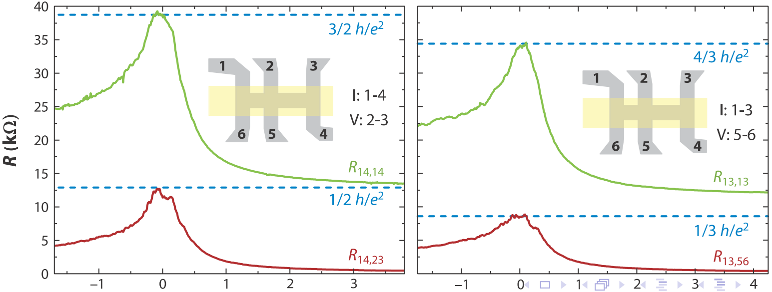
- The four-terminal resistance is

$$\begin{aligned} R_{13,56} &\equiv \frac{V_{56}}{I_{13}} \\ &= \frac{h}{3e^2} \end{aligned}$$

Comparison of Theory and Experiment

- **Summary of the different two- and four-terminal resistances**

$R_{ij,k\ell}$	Expected value
$R_{14,14}$	$3h/2e^2$
$R_{14,23}$	$h/2e^2$
$R_{13,13}$	$4h/3e^2$
$R_{13,56}$	$h/3e^2$



"Helical" Liquid

- Edge Hamiltonian

$$H = \int \frac{dk}{2\pi} \left(\psi^\dagger_{k,\uparrow} v k \psi_{k,\uparrow} - \psi^\dagger_{k,\downarrow} v k \psi_{k,\downarrow} \right) + H_{\text{pert}}$$

- Time-Reversal Symmetry (TRS) expressed as

$$T^{-1}\psi_{k,\uparrow}T = \psi_{-k,\downarrow}, \quad T^{-1}\psi_{k,\downarrow}T = -\psi_{-k,\uparrow}$$

- If H_{pert} does not respect TRS a simple "mass" term can be added

$$H_{\text{mass}} = \int \frac{dk}{2\pi} m \left(\psi^\dagger_{k,\uparrow}\psi_{k,\downarrow} + \text{h.c.} \right)$$

  but $T^{-1}H_{\text{mass}}T = -H_{\text{mass}}$
- If H_{pert} respects TRS then it can only include $2n$ particle scattering processes like $\psi^\dagger_{k,\uparrow}\psi^\dagger_{k',\uparrow}\psi_{p,\downarrow}\psi_{p',\downarrow}$

$\mathbb{Z}_2$ versus $\mathbb{Z}$

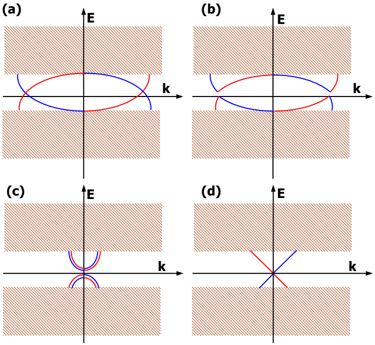
- In a topologically non-trivial system there must be odd number of Kramers' pairs crossing the Fermi energy

- Say we had two copies of *helical* edge states in TRS system

$$H = \int \frac{dk}{2\pi} \sum_{s=1,2} \left(\psi^\dagger_{k,s,\uparrow} v_s k \psi_{k,s,\uparrow} - \psi^\dagger_{k,s,\downarrow} v_s k \psi_{k,s,\downarrow} \right)$$

- $H_{\text{pert}} = \psi^\dagger_{k,s,\uparrow}\psi_{-k,s',\downarrow}$ is allowed only for $s \neq s'$ and the follwing mass term can gap out the system

$$\int \frac{dk}{2\pi} M \left(\psi^\dagger_{k,s,\uparrow}\psi_{k,s',\downarrow} + \text{h.c.} \right)$$



Three-Dimensional Topological Insulators

- Similar to HgTe band inversion predicted in alloy Bi_xSb_{1-x}
- Topological Band Theory (TBT) worked out by Fu, Kane, and Mele
- New topological invariant: $(\nu_0; \nu_1\nu_2\nu_3)$ for *inversion symmetric* materials
- Bulk defects causes Fermi level to lie in the middle of a band, i.e. cannot detect edge states using transport
- Angle-Resolved Photoemission Spectroscopy (ARPES) can independently image surface and bulk spectrum

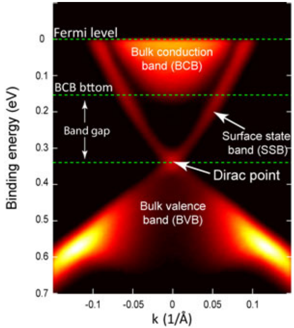


Figure : ARPES on Bi_2Se_3. Courtesy of Fisher Group (Stanford)

Conclusion and Outlook

- **Experimental Work**
  - So far experiments have only confirmed the existence of edge states in 2-D and 3-D topological insulators
  - However, the intrinsic topological properties like
    - Fractional charge for the quantum spin Hall
    - Topological Kerr/Faraday effect
    - Topological magneto-electric effect
    - Monopole effect for 3-D topological insulators

    have not been observed yet
  - Moreover, not all systems possessing topological order necessarily have edge states
- **Theoretical Investigations**
  - What new types of exotic phases are possible by introducing interaction?
  - Topological ____________ insulator

- Anderson
- Band
- Crystalline
- Crystalline
- Kondo
- Mott

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